Analysis Qualifying Exam
September 20, 2008

1. Let μ_F be the Lebesgue-Stieltjes measure associated to the function

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x. \end{cases}$$

Prove that μ_F is absolutely continuous with respect to the Lebesgue measure m and find the Radon-Nikodym derivative of μ_F with respect to m.

2. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0),\ |x| \le 1,\ |y| \le 1 \\ (0,0) & \text{if } x = y = 0. \end{cases}$$

Show that both iterated integrals of f coincide and have value 0, but that f is not integrable on $[-1,1] \times [-1,1]$ (hint: try a change of coordinates). What hypothesis of Fubini's Theorem fails here?

3. Let μ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \to \mathbb{R}$. Prove that f_n converge uniformly to a function f if and only if f_n converge to f in measure.

4. Let $Tf(x) = x^{-1/2} \displaystyle\int_0^x f(t)\,dt$.

   (a) Prove that T is a bounded linear operator from $L^2(0,\infty)$ to $L^\infty(0,\infty)$.

   (b) Prove that $Tf \in C_0(0,\infty)$. Recall that $C_0(0,\infty)$ is the set of continuous functions on $(0,\infty)$ such that $\lim_{x\to 0} g(x) = 0 = \lim_{x\to\infty} g(x)$.

   Hint: Simple functions!

5. Let μ be a Borel measure on $\mathbb{R}$, finite over compact sets and with $\mu((0,1]) = 1$. Prove that if for every $s \in \mathbb{R}$, $\mu(s+E) = \mu(E)$, then μ is the Lebesgue measure on $\mathbb{R}$. *Hint:* Start by calculating $\mu(a,b]$ when $a, b \in \mathbb{Q}$.

6. On the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ we define μ^* as follows: for every $E \in \mathcal{P}(\mathbb{N})$,

$$\mu^*(E) = \begin{cases} \frac{n}{n+1} & \text{if } \operatorname{card}(E) = n \\ 1 & \text{if } \operatorname{card}(E) = \infty. \end{cases}$$

   (a) Prove that for every $A, B \in \mathcal{P}(\mathbb{N})$, $\mu^*(A) + \mu^*(B) \ge 1$.

   (b) Prove that μ^* is an outer measure.

   (c) Describe the σ-algebra of measurable sets.