Analysis Preliminary Examination
Spring 2012

Unless a problem states otherwise m will denote Lebesgue measure, m^* will denote Lebesgue outer measure, and $\mathcal{L}$ will denote the Lebesgue measurable sets. Please justify your answers.

Part I: Measure Theory: Complete **five** of the following six questions.

I.1 Let μ^* be an outer measure on the set X.

a) State what it means for a set $A \subseteq X$ to be μ^*-measurable.

b) If we let $\mathcal{M}$ denote the set of μ^*-measurable sets, show that $\mathcal{M}$ is closed with respect to complements and finite unions.

c) Show that if $\mu^*(A) = 0$ then $A \in \mathcal{M}$.

I.2 Let $\{f_n\}$ be a sequence of positive valued measurable functions on $\mathbb{R}$.

a) State the Monotone Convergence Theorem.

b) If we define $f = \sum f_n$ use the Monotone Convergence Theorem to show that $\int f\, dm = \sum \int f_n\, dm$.

I.3 Let $F : \mathbb{R} \to \mathbb{R}$ be bounded and increasing.

a) Prove that F is of bounded variation.

b) Determine $T_F(x)$ where T_F is the total variation function of F.

I.4. a) Give the defintion of a measurable function $f : E \to \mathbb{R}$, where E is a measurable subset of $\mathbb{R}$.

b) Show that if f an extended real-valued measurable function and g is a continuous function on $(-\infty, \infty)$ then $g \circ f$ is measurable.

c) If g is also a measurable function on $(-\infty, \infty)$, would it follow that $g \circ f$ is a measurable function? Yes or No.

I.5. a) Show that a measurable function f is integrable function over a measurable set E if and only if $|f|$ is integrable over E.

b) Is it possible to have a non-integrable function f with $|f|$ is integrable? If so, provide an example.

I.6. Let $f : X \to \mathbb{R}$ be a μ-integrable function and $g : Y \to \mathbb{R}$ be a ν-integrable function. If $h(x, y) = f(x)g(y)$ for every $x \in X$ and $y \in Y$, show that

$$\int_{X\times Y} h\ d\mu \times d\nu = (\int_X f d\mu)(\int_Y g d\nu).$$

Part II: Complex and Functional Analysis: Complete **three** of the following six questions. Be sure to clearly mark which problems you want graded for this section.

II.1 Consider the measure space $([0,1], \mathcal{L}, m)$.

a) State Hölder's inequality.

b) Use Hölder's inequality to prove Minkowski's inequality for $1 < p < \infty$.

II.2 Let $T \in B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Prove that $\|T\|^2 = \|T^*T\| = \|T^*\|^2$.

II.3 Let X be a Banach space, and $T : X \to Y$ be bijective.

a) State the Open Mapping Theorem.

b) Use the Open Mapping Theorem to prove that if T is continuous then so is T^{-1}.

II.4 Let $f = u+iv$ be a holomorphic function on a domain Ω. Prove that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ on Ω.

II.5 Let f be holomorphic on a bounded domain Ω and continuous on $\overline{\Omega}$.

a) State the maximum modulus theorem for f.

b) Show that if $|f(z)| \geq \max\{|f(w)| : w \in \partial\Omega\}$ for some $z \in \Omega$ then f is a constant.

II.6 Let f be a function defined on the open unit disc such that $f(\frac{1}{n}e^{2\pi \frac{i}{n}}) = 0$ for each n and $f(-\frac{i}{2}) = 1$ explain why f is not holomorphic.