Analysis Qualifying Exam
January 2019

Submit six of the problems from part 1, and three of the problems from part 2. Start every problem on a new page, label your pages and write your student ID on each page.

1 Real Analysis

Throughout, m denotes Lebesgue measure. You may use without proof the Lebesgue dominated convergence theorem and Fubini's theorem.

1. Show: If $f_n \to f$ in $L^1(\mu)$, then $f_n \to f$ in measure.
2. Let μ_0 be a premeasure on an algebra $\mathcal{A}$, and let μ^* be the induced outer measure. Denote by $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$. Let E be μ^*-measurable. Show that there exists $B \subset E$ such that B is a countable intersection of sets from $\mathcal{A}_\sigma$ and $\mu^*(E \backslash B) = 0$. (Hint: Use the definition of outer measure.)
3. Assume $f : \mathbb{R} \to \mathbb{C}$ is differentiable on $\mathbb{R}$ and has a bounded derivative.
   (a) Show that f has bounded variation on $[a, b]$ for any finite $a < b$.
   (b) Must f be of bounded variation on $\mathbb{R}$? (Justify.)
4. Let f be real-valued.
   (a) Give the definition of a measurable function.
   (b) If f is measurable, is f^2 measurable? Justify.
   (c) If f^2 is measurable, is f measurable? Justify.
5. Define $F : \mathbb{R} \to \mathbb{R}$ by
$$F(x) = \begin{cases} 0 & \text{if } x < 1, \\ 2 - x & \text{if } 1 \le x < 2, \\ 0 & \text{if } 2 \le x. \end{cases}$$
   Let μ_F be the (signed!) Borel measure whose distribution function is F.
   (a) Calculate $\mu_F((0, 1))$ and $\mu_F((0, 1])$.
   (b) Find the distribution function of the total variation measure $|\mu_F|$.
6. Let $[a, b] \subset \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a bounded Borel measurable function. Prove that f is Riemann integrable if and only if it is continuous almost everywhere.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Lebesgue measureable, such that
$$\lim_{h \to 0} \int_0^1 \frac{|f(x + h) - f(x)|}{h} dx = 0.$$
   Show that f is a.e. constant on the interval $[0, 1]$.

8. Prove Egorov's theorem: Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of measurable functions which converge almost everywhere to a measurable function $f : [0,1] \to \mathbb{R}$. Then for every $\epsilon > 0$ there exists a measurable set $E \subset [0,1]$ with measure $|E| < \epsilon$ such that f_n converges to f uniformly on $[0,1] \setminus E$.

   Hint: Consider the sets $E_m(k) = \cap_{m \geq n} \{x : |f_m(x) - f(x)| < \frac{1}{k}\}$.

9. Let μ be a finite measure that is singular with respect to Lebesgue measure. Show that

$$\lim_{\epsilon \to 0} \frac{\mu([x-\epsilon, x+\epsilon])}{2\epsilon} = 0.$$

   Hint: Lebesgue differentiation.

2 Complex and Functional Analysis

10. Assume $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C} \backslash [-1,1]$ and continuous on all of $\mathbb{C}$. Prove that f is entire.

11. (a) State the maximum modulus theorem.

    (b) Let f be a holomorphic function mapping the unit disk into the unit circle. Show that f is constant.

12. Let f be holomorphic on $\mathbb{C}$ and assume that there exists $c > 0$ such that

$$|f(z)| \leq c(1 + \sqrt{|z|})^5$$

    for all complex z. Show that f is a polynomial. What can you say about the degree of this polynomial?

13. Let $(X, \| \, \|)$ be a normed space. Prove that if every linear functional on X is continuous, then X is finite dimensional.

14. (a) State Riesz Representation Theorem on Hilbert spaces.

    (b) Use Riesz Representation Theorem to show that for every Hilbert space H, H is isometrically isomorphic to H^*.

15. (a) State Hahn-Banach Theorem.

    (b) Let $(X, \| \, \|)$ be a normed space, $x_0 \in X,\ x_0 \neq 0$. Prove that $\exists f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.