North Dakota State University

MATH 784

Prelim August 2023

- **Answer any 6 problems. If you attempt all the problems, clearly indicate which 6 you want to be graded. Otherwise the first 6 will be graded.**

Problem 1. *Let D be a bounded domain in $\mathbb{R}^n$ and $u(x)$ is defined on D. Show that for a given function $f : D \to \mathbb{R}$ a solution of the boundary value problem*

$$\begin{cases} \Delta(\Delta u) = f, & \text{in } D \\ u = \Delta u = 0, & \text{on } \partial D, \end{cases}$$

is unique (provided it exists).

Problem 2. *Let $u(x) \geqslant 0$ be continuous in closed bounded domain $\bar{D} \subset \mathbb{R}^n$ and Δu is continuous in $\bar{D}$. Suppose that*

$$\Delta u = u^2, \quad u\big|_{\partial D} = 0.$$

Prove that $u \equiv 0$, in D. What can you say about $u(x)$ when the condition $u(x) \geqslant 0$ in D is dropped?

Problem 3. *Write down an explicit formula for a function $u(x,t)$ solving*

$$\begin{cases} u_t + b \cdot Du + cu = \Delta u, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x), & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are constants.

Problem 4. *Assume u is a harmonic function in an open bounded set U. Prove that the function $v = \|Du\|^2$ satisfies inequality $-\Delta v \leq 0$ in U.*

Problem 5. *Consider one-dimensional problem:*

$$u_t = u_{xx}, \quad x > 0, \quad t > 0,$$

$$u(x,0) = g(x), \quad x > 0$$
$$u(0,t) = 0, \quad t > 0,$$

where g is continuous and bounded for $x \geqslant 0$ and $g(0) = 0$. Find a formula for the solution $u(x,t)$.

Problem 6. *Let*

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu,$$

where L is uniformly elliptic with smooth coefficients. Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided $c(x) \geqslant -\mu$, for $x \in U$.

Problem 7. *Verify that if $n > 1$, the unbounded function*

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(U)$, where U is an open unit ball in $\mathbb{R}^n$ with center at the origin.