Problems for Preliminary Exam  
Applied Mathematics  
August 2022

Ordinary Differential Equations

1. Show that every integral curve of  
   \[
   \dot{x} = \sqrt{\frac{x^2 + 1}{t^4 + 1}}, \quad x(t) \in \mathbb{R},
   \]
   has two horizontal asymptotes.

2. Is there a bounded solution to (here \(x(t) \in \mathbb{R}\))  
   \[
   \dot{x} - x = \cos t - \sin t?
   \]

3. Let  
   \[
   \dot{x} = f(x), \quad x(t) \in \mathbb{R}^n, \quad f \in C^1, \quad f: \mathbb{R}^n \to \mathbb{R}^n, \quad f(\dot{x}) = 0, \quad \dot{x} \in \mathbb{R}^n.
   \]
   Give a definition for \(\dot{x}\) to be Lyapunov stable, asymptotically stable, or unstable. Formulate the definition for a differentiable \(V: \mathbb{R}^n \to \mathbb{R}\) to be a Lyapunov function for \(\dot{x}\), a strict Lyapunov function for \(\dot{x}\).
   Formulate and prove Lyapunov’s theorem on (Lyapunov, asymptotic) stability of \(\dot{x}\) by direct Lyapunov method (i.e., by assuming existence of a (strict) Lyapunov function).

4. Sketch the phase portrait of  
   \[
   \dot{x} = 1 + 2 \sin x, \quad x(t) \in \mathbb{R}.
   \]

5. For which \(\alpha\) the trivial equilibrium of  
   \[
   \begin{align*}
   \dot{x}_1 &= \alpha x_1 - x_2, \\
   \dot{x}_2 &= \alpha x_2 - x_3, \\
   \dot{x}_3 &= \alpha x_3 - x_1,
   \end{align*}
   \]
   is Lyapunov stable, asymptotically stable, unstable?

6. Find Green’s function for  
   \[
   y'' + y = f(x), \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).
   \]

7. Show that the problem  
   \[
   \dot{x} = 2 \sqrt{t x}, \quad x(0) = 0
   \]
   has more than one solution passing through the origin. For the full credit describe all the solution to this problem.  
   Which condition(s) of the existence and uniqueness theorem fail?