Problems for Preliminary Exam
Applied Mathematics
August 2022

Partial Differential Equations

1. Consider the following problem:

$$
\begin{aligned}
u_{tt} &= u_{xx}, \quad x>0, \quad t>0,\\
u(0,x) &= f(x-2), \quad x>0,\\
u_t(0,x) &= 0, \quad x>0,\\
u(t,0) &= 0, \quad t>0,
\end{aligned}
$$

where

$$
f(x)=\begin{cases} 0, & x<-1,\\ x+1, & -1\le x\le 0,\\ 1-x, & 0<x\le 1,\\ 0, & x>1. \end{cases}
$$

Sketch the solution to this problem at different (representative) time moments and justify your sketches (you are not asked to actually compute the analytical formulas for the solution).

2. Let

$$
\Omega = \{(x,y)\in\mathbb{R}^2 \colon x^2+y^2<1,\, y>0\}.
$$

Solve

$$
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega,\\
u(r,0) &= u(r,\pi)=0,\\
u(1,\theta) &= \theta(\theta-\pi),
\end{aligned}
$$

where r,θ are polar coordinates on the plane.

What is

$$
\max_{\overline{\Omega}} u?
$$

Here, as usual, $\overline{\Omega}=\Omega\cup\partial\Omega$.

3. Find all u harmonic in $\mathbb{R}^2$ for which

$$
u_x(x,y) < u_y(x,y)
$$

for all $(x,y)\in\mathbb{R}^2$.

4. Give a definition of a well-posed problem. Show that the following problem is not well posed.

Let u be the solution to the problem

$$
\begin{aligned}
u_t &= u_{xx}, \quad 0<x<\pi, \quad 0<t<T,\\
u(x,T) &= g(x),\\
u(0,x) &= u(\pi,x)=0.
\end{aligned}
$$

1. Make the variable change $t = T - s$.

2. Solve (formally) the obtained initial–boundary value problem by separation of variables. Specify which hypotheses on g guarantee that the expression you found is indeed a solution.

3. Show the solution does not depend continuously on the initial data, by taking the sequence of the problems with $g_n(x) = \frac{1}{n}\sin nx$. (*Hint:* We have here that $g_n \to 0$ uniformly, you need to show that the corresponding solutions at time T do not converge uniformly to 0.)

5. Write down the integral representation of the solution to the heat equation $u_t = u_{xx}$, $x \in \mathbb{R}$, $t > 0$ with the initial condition $u(0, x) = x^2$. Find this solution explicitly.

Hint: You may do it directly, by computing the required integral, which is, however, time consuming and prone to arithmetic mistakes. Instead you may first show that $v = u_{xxx}$ solves the heat equation with the zero initial condition and therefore (why?) $u(t, x) = A(t)x^2 + B(t)x + C(t)$, where A, B, C can be found by direct substitution.

6. Solve

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$

with the initial condition

$$u(0, x) = -x.$$

Sketch the (projected) characteristics.

For which values of t the classical solution to this problem exists?

7. Consider the wave equation in $\mathbb{R}^3$:

$$u_{tt} = c^2 \Delta u, \quad x \in \mathbb{R}^3, \quad t \in (0, \infty),$$

with the initial conditions $u(x, 0) = g(x), u_t(x, 0) = h(x)$. Suppose that g, h have support contained in $B_\rho(0)$ (i.e., in the ball of radius ρ with the center at the origin). Describe the support of the solution at each time moment $t > 0$.