Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many injective functions from [8] to [12] have 2 in their image, but not 3?
2. The following is a permutation written in one-line notation. Rewrite it in cycle notation.

$$389765241$$

3. Find the generating function encoding the sequence, $\{0, 2, 0, 4, 0, 6, 0, 8, \ldots\}$.
4. 25 people are in line for food today and again tomorrow. Assuming that the order of the line is totally random each day, what is the probability that exactly one person is in the same position in line today and tomorrow?
5. Give a combinatorial proof of the identity,

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}.$$

Longer questions: (10 points each)

6. A university wants to make a committee of 7 people from 10 physicists, 8 mathematicians, and 8 chemists. How many ways can a committee be formed that includes at least one member from each of the three departments? (In this problem, each physicist is distinct, each mathematician is distinct, and each chemist is distinct.)
7. Give a combinatorial interpretation of the following generating function coefficient, and then use your interpretation to compute the numerical value of the coefficient. (No credit will be awarded for expanding the generating function as a series.)

$$[x^{20}]\left(\frac{1}{1-x}\right)^2 \cdot \frac{x^2}{1-x}$$

8. Let P_n equal the number of rooted plane trees with n vertices where the root has degree at least 2. Prove that $P_n = C_{n-1} - C_{n-2}$, where C_n is defined to be the number of Dyck paths with $2n$ steps. (Assume no other facts about C_n.)
9. Let $p(k, \ell)$ be the partitions with at most k parts, each of size at most ℓ. Derive an explicit formula for $p(k, \ell)$ by considering the Ferrers diagrams of such partitions.
10. A combinatorics class has 20 distinct students, and they will break up into 4 groups of size 1 or more for the final project. If Student 1 wants to work with at least 2 other people, how many ways can groups be formed? Your answer may include summations.