Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many permutations of length 5 start with 32 when written in one-line notation and avoid the pattern 123?
2. You have n distinct friends and n distinct books. Each friend dislikes exactly one book, and no two friends dislike the same book. If you distribute one book to each friend randomly, what is the probability that nobody dislikes the book they receive?
3. In how many ways can the letters AAABBCDE be arranged so that no two As are adjacent?
4. Find the closed form of the generating function encoding the sequence,
$$\{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}.$$
5. I have 200 identical red stickers and 100 identical blue stickers, and I want to distribute them to three distinct people. At the end of the day, I only care about how many red and blue stickers each person received. How many ways can I distribute the stickers?

Longer questions: (10 points each)

6. You are creating a committee of 15 scientists out of a group of 100 biologists, 200 physicists, 300 chemists, and 400 mathematicians. How many ways can you create a committee that has at least one biologist, at least one physicist, at least one chemist, and any number of mathematicians?
7. How many partitions are there with at most 100 parts where each part is of size at most 90? Justify your answer completely.
8. Let B_n be the number of set partitions of $\{1, 2, \ldots, n\}$. Give a combinatorial proof of the identity,
$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$
9. How many pairs of lattice paths (A, B) are there where path A starts at $(0, 2)$ and ends at $(5, 10)$, path B starts at $(2, 0)$ and ends at $(7, 8)$, and A and B never intersect?
10. Recall that a permutation σ can be represented in a matrix P_σ where the entry (i, j) of P_σ is one if and only if $\sigma(i) = j$, and it is zero otherwise. Define the composition of permutations $\sigma \circ \tau$ by $(\sigma \circ \tau)(i) = \sigma(\tau(i))$. How does the matrix $P_{\sigma\circ\tau}$ relate to the matrices P_σ and P_τ? Prove your answer.