Instructions. Answer any 4 short questions, and any 4 long questions. Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many monomials of degree 7 are there in the variables $x_1, x_2, x_3, x_4,$ and $x_5$?

2. How many multisets are contained within the multiset \{A, A, A, B, B, C, D, E, F\}?

3. How many distinct permutations are there on [10] that have exactly two cycles when written in cycle notation?

4. How many ways can 6 days be picked from January (which has 31 days) so that no two picked days are consecutive?

5. Label the vertices of a hexagon with 1 through 6. How many ways can the hexagon be triangulated? Your answer should be a single number.

Longer questions: (10 points each)

6. How many lattice paths from (0, 0) to (n, n) stay weakly above the line $y = x − 1$? Derive your answer from scratch.

7. Give a combinatorial proof of the following identity:

$$\binom{n+3}{5} = \sum_{k=2}^{n} \binom{k}{2} \binom{n+2-k}{2}$$

8. Consider a set $S$ of objects, and a list of properties $c_1, c_2, \ldots, c_t$.

- Let $S_1 = \sum_i N(c_i), S_2 = \sum_{1 \leq i < j \leq t} N(c_i c_j)$, and so on (where $N(c_2 c_3)$ is the number of elements in $S$ satisfying $c_2$ and $c_3$, for example).

- Let $E_m$ be the number of elements in $S$ that satisfy exactly $m$ of the properties in $S$.

Prove the following identity holds for each $m$ between 1 and $t$:

$$E_m = \sum_{j=m}^{t} (-1)^{j-m} \binom{j}{m} S_j$$

The following identity may be useful: $\binom{j}{m} \binom{t}{j} = \binom{t}{m} \binom{t-m}{j-m}$.

9. Use generating functions to prove that the number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts. In your proof, it should be clear how your generating functions were derived.

10. How many partitions have a Young diagram that fits into a $6 \times 6$ square and a first part that is at least 5?