Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many monomials of degree 7 are there in the variables x_1, x_2, x_3, x_4, and x_5?
2. How many multisets are contained within the multiset $\{A, A, A, B, B, C, D, E, F\}$?
3. How many distinct permutations are there on $[10]$ that have exactly two cycles when written in cycle notation?
4. How many ways can 6 days be picked from January (which has 31 days) so that no two picked days are consecutive?
5. Label the vertices of a hexagon with 1 through 6. How many ways can the hexagon be triangulated? Your answer should be a single number.

Longer questions: (10 points each)

6. How many lattice paths from $(0, 0)$ to (n, n) stay weakly above the line $y = x - 1$? Derive your answer from scratch.
7. Give a combinatorial proof of the following identity:
$$\binom{n+3}{5} = \sum_{k=2}^{n} \binom{k}{2}\binom{n+2-k}{2}$$
8. Consider a set S of objects, and a list of properties $c_1, c_2, \ldots, c_t$.
   - Let $S_1 = \sum_i N(c_i), S_2 = \sum_{1\le i<j\le t} N(c_i c_j)$, and so on (where $N(c_2c_3)$ is the number of elements in S satisfying c_2 and c_3, for example).
   - Let E_m be the number of elements in S that satisfy exactly m of the properties in S.

   Prove the following identity holds for each m between 1 and t:
$$E_m = \sum_{j=m}^{t} (-1)^{j-m} \binom{j}{m} S_j$$
   The following identity may be useful: $\binom{j}{m}\binom{\ell}{j} = \binom{\ell}{m}\binom{\ell-m}{j-m}$.
9. Use generating functions to prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. In your proof, it should be clear how your generating functions were derived.
10. How many partitions have a Young diagram that fits into a 6×6 square and a first part that is at least 5?