Math 747 Preliminary Exam

August 2024

Note: "Manifold" refers to a manifold **without** boundary.

Problem 1. Prove that the circle $\mathbb{S}^1$, the closed interval $I = [0, 1]$, and the real line $\mathbb{R}$ are pairwise non-homeomorphic.

Problem 2. Let $\omega = (y^2 + x)dx \wedge dy - yzdy \wedge dz + (z + y)dz \wedge dx$ be a differential 2-form in $\mathbb{R}^3$.

a) Is ω exact? Is it closed?

b) Is there a differential 1-form η such that $d\eta \wedge \eta = d\omega$?

Problem 3. Find all real values of c such that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given as

$$F(x, y) = (4y - 5x, 2x + cy^2)$$

is

a) an immersion;

b) an embedding;

c) a topological embedding.

Problem 4. a) Define the notion of a degree of a map.

b) Find explicitly a smooth map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ of degree 9.

c) For any given $d \in \mathbb{Z}$ and $n \in \mathbb{N}\backslash\{0\}$, find explicitly a smooth map $g : \mathbb{T}^n \to \mathbb{T}^n$ of degree d.

Problem 5. a) State Sard's Theorem.

c) Prove Sard's Theorem in the one-dimensional case (i.e. for the maps $f : M \to N$ where M, N are 1-dimensional smooth manifolds).