NDSU Mathematics Department
GEOMETRY and TOPOLOGY QUALIFYING EXAMINATION
August 21, 2019

- Please write all answers using a blue pen. Only write on one side of the paper. Clearly cross-out any work which you do NOT want us to grade.
- Unless otherwise stated, "manifold" refers to a manifold without boundary.

Part A

A1 Let X be a quotient space of the Klein bottle obtained by identifying two distinct points. Compute the fundamental group and all the homology groups of X.

A2 Let $\mathbb{Z}_6$ act on $S^3 = \{(z, w) \in \mathbb{C}^2,\ |z|^2 + |w|^2 = 1\}$ via $(z, w) \mapsto (\zeta z, \zeta w)$ where ζ is a sixth root of unity. Denote by L the quotient $S^3/\mathbb{Z}_6$.

(1) What is the fundamental group of L?
(2) Describe all coverings of L.
(3) Show that any continuous map $L \to S^1$ is nullhomotopic.

A3 Let D^3 be the 3-dimensional closed unit ball. Let $F : D^3 \to D^3$ be a continuous map. Show that f has a fixed point.

A4 Let X and Y be topological spaces and $f : X \to Y$ a map which is continuous and bijective.

(1) Give an example showing that f need not be a homeomorphism. Be sure to prove that the map in your example is continuous and not a homeomorphism.
(2) Prove that f must be a homeomorphism under the additional assumptions that X is compact and Y is Hausdorff.

A5 Let S^n, $n \geq 1$, be the unit sphere in $\mathbb{R}^{n+1}$ given by $||x|| = 1$. Suppose that $f : S^n \to S^n$ is a smooth map such that the position vector of $x \in S^n$ and the position vector of $f(x) \in S^n$ are perpendicular to each other for all $x \in S^n$.

(1) Prove that f is homotopic to the identity map of S^n
(2) Show that such an f exists if and only if n is odd.

Part B

B1 Let M be a nonempty topological n-manifold without boundary with $n \geq 1$. If M has a smooth structure, show that M has uncountably many smooth structures

B2 Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\Psi(x, y) = x^2 - y^2$.

(1) Show that $\Psi^{-1}(0)$ is **not** an embedded submanifold of $\mathbb{R}^2$.

(2) Can $\Psi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?

B3 Let M be the smooth submanifold of $\mathbb{R}^3$ defined by

$$(1 - z^2)(x^2 + y^2) = 1.$$

(1) Define a vectorfield on $\mathbb{R}^3$ by

$$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1 - z^2)\frac{\partial}{\partial z}.$$

Show that the restriction of V to M is a tangent vector field to M.

(2) Show that the family of maps

$$\phi_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z)$$

restricts to a one-parameter family of diffeomorphisms of M. For each t, determine the vector field $d\phi_t(V)$ on M.

B4 Let $Z \subset \mathbb{R}^2$ be the unit circle and consider the map $f : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}^2$ given by

$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

(1) Show that f is not transverse to Z.

(2) Find a smooth homotopy $F : [0, 1] \times \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}^2$ such that $F(0, x) = f(x)$ and $F(1, x)$ is transverse to Z. Justify your claim.

B5 Let X be a compact connected orientable n-manifold without boundary and let $\omega \in \Omega^{n-1}(X)$. Prove that $d\omega$ must vanish at some point of X.