NDSU MATHEMATICS DEPARTMENT
Geometry and Topology Qualifying Exam. May 15th, 2017.

Unless otherwise stated "manifold" refers to a manifold **without** boundary.

Problem 1. Let X be a Hausdorff topological space and let $F, K \subseteq X$ be disjoint compact subsets. Show that there exist open sets $U, V \subseteq X$ such that $F \subseteq U, K \subseteq V$ and $U \cap V = \emptyset$.

Problem 2. Let $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ be the torus and let $S = \mathbb{S}^1 \times \{0\}$ be a fixed circle in $\mathbb{T}$. Fix an integer $k \geq 1$, and let $\mathbb{T}_1, \ldots, \mathbb{T}_k$ be disjoint homeomorphic copies of $\mathbb{T}$, with $S_1, \ldots, S_k$ be the corresponding homeomorphic copies of S. Compute the fundamental group of the space obtained from the union of tori $\mathbb{T}_1, \ldots, \mathbb{T}_k$ by identifying the circles $S_1, \ldots, S_k$.

Problem 3. What compact 2-manifolds can one obtain by pairwise gluing the sides of a regular hexagon?

Problem 4. Let X be the complement of a tame knot in $\mathbb{R}^3$. Show that $H_1(X) \cong \mathbb{Z}$.

HINT: Recall that a tame knot $K \subseteq \mathbb{R}^3$ can be "thickened up" - it admits an extension to an embedding of a solid torus. More precisely, there exists a solid torus $\mathbb{S}^1 \times D^2$ embedded in $\mathbb{R}^3$ such that $K = \mathbb{S}^1 \times \{0\}$.

Problem 5. Let

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \cup \{(0, 0, z) : -1 \leq z \leq 1\}$$

viewed as a subspace of $\mathbb{R}^3$. Let also $Y = \mathbb{S}^1 \vee \mathbb{S}^2$.

a) Are X and Y homotopically equivalent?

b) Describe the universal covering space of Y.

Problem 6. Let M, N be smooth manifolds, and suppose $F : M \to N$ is a smooth injective immersion.

a) Prove that if M is compact, then F is an embedding.

b) Give an example of a smooth injective immersion that is not an embedding.

Problem 7. a) State the Inverse Function Theorem.

b) Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$H(x, y) = (ye^x, 2x).$$

Find the set of points at which H is regular. Is H a diffeomorphism, a local diffeomorphism or neither?

Problem 8. Define a 1-form on the punctured plane $\mathbb{R}^2 \backslash \{0\}$ by

$$\omega = -\left(\frac{y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

Calculate $\int_C \omega$ for the circle C of radius r centered on the origin.

Problem 9. Define

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$$

as a 2-form on $\mathbb{R}^6$. Show that no diffeomorphism $\varphi : \mathbb{R}^6 \to \mathbb{R}^6$ which satisfies $\varphi^*\omega = \omega$ can map the unit sphere S^5 to a sphere of radius $r \neq 1$. (Hint: Consider $\omega \wedge \omega \wedge \omega$.)

Problem 10. Identify $M(2, \mathbb{R})$, the set of two-by-two matrices with real entries, with $\mathbb{R}^4$. Let $SL(2, \mathbb{R}) \subset M(2, \mathbb{R})$ be the set of matrices with determinant one.

a) Show that $SL(2, \mathbb{R})$ is a smooth submanifold of $M(2, \mathbb{R})$.

b) Calculate the tangent space $T_{Id}SL(2, \mathbb{R})$ as a subspace of $T_{Id}M(2, \mathbb{R})$. (Hint: Consider a curve $\alpha(t) \in SL(2, \mathbb{R})$ with $\alpha(0) = Id$.)

c) Is the line

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

for $t \in \mathbb{R}$ transverse to $SL(2, \mathbb{R})$? Justify your answer.