TREE-LEVEL SCATTERING OF MASSIVE FERMIONS BY A MASSLESS ANTISYMMETRIC TENSOR FIELD

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ABSTRACT

Lu, Bin, M.S., Department of Physics, College of Science and Mathematics, North Dakota State University, June 2002. Tree-Level Scattering of Massive Fermions by a Massless Antisymmetric Tensor Field. Major Professor: Dr. Patrick F. Kelly.

As part of a potential extension to the standard model of particle physics, an interaction between an antisymmetric tensor field and fermion fields is considered. Processes involving tree-level scattering of massive fermions by exchange of a massless antisymmetric tensor are examined. Computation of scattering amplitudes, differential cross section, and total cross section may prove relevant and useful for experimentalists.

ACKNOWLEDGMENTS

I would like to thank Dr. Patrick Kelly and Terry Pilling for devoting the long hours teaching me everything I know in this thesis. Thanks to Feng Hong and his family for three years of hospitality. Many thanks to Master Marquart and my kumdo classmates for giving me the discipline and strength to make this possible. The lessons shall benefit me for the rest of my life. My family's belief in me never wavered when I was having doubts about myself. All thanks go to my parents, sisters, and Aunt Delan. And to D.L. and L.W.S., if I never met you, I probably would have finished this thesis sooner. Then again, I will not exchange the memory for anything in this world.

In memory of Jim, Drew, and Wolly; we miss you.

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CHAPTER 1

INTRODUCTION

The standard model is currently the accepted theory of fundamental particles. Particles in the standard model can be categorized as either fermions or bosons. Fermions have odd-half-integer $\{\frac{1}{2}, \frac{3}{2}, \ldots\}$ spins. All the matter we see around us is comprised of fermions. The forces of interaction are carried by bosons which have integer spins. The standard model is founded upon the principle of gauge invariance; i.e., particle interactions are consequences of gauge symmetries. For an introduction to and review of the standard model, see [1, 2, 3, 4, 5, 6, 7, 8, 9] and references therein.

It is worthwhile to consider extensions of the standard model. For example, string theory suggests that the building blocks of nature are actually distinct states of fundamental strings [10, 11]. The extension of the standard model that we wish to consider in this thesis is the introduction of an antisymmetric tensor (2-form) field. The idea arises naturally from string theory [10, 11, 12], supergravity [13, 14], and Einstein-Cartan relativity [15, 16, 17].

We will compute the effects of antisymmetric tensor particle exchange between two physical fermions. Using the Born Approximation, we examine its lowest-order effect: one antisymmetric tensor particle is exchanged, and no other interactions are present. First, we look at the case in which the (massive) fermions are identical. Second, we consider the case where the external fermions are distinct and have unequal masses.

CHAPTER 2

TECHNICAL PRELIMINARIES

In this chapter, we will display the Lagrangian and extract the relevant Feynman rules. We will also list the definitions and tools necessary for our calculation.

2.1. Notation and conventions

We briefly describe the notation and conventions in use throughout this thesis. More details can be found among the standard references [1, 2, 4, 5, 6, 8]. Spacetime is four dimensional; vectors in spacetime have four components. The energy-momentum four-vector is denoted

$$\begin{array}{ll} \text{contravariant representation} & p^\mu = (E, \vec{p}) = (E, p^x, p^y, p^z) \\ \text{covariant representation} & p_\mu = (E, -\vec{p}) = (E, -p^x, -p^y, -p^z) \ , \end{array} \tag{2.1}$$

where the "zeroth" component is the total relativistic energy and $\vec{p}$ represents the three spatial components of the relativistic momentum. In keeping with conventional usage, we are using *natural units* in which the speed of light is unitary; $c = 1$. Otherwise, explicit factors of c would multiply the spatial components in the above expressions. The squared magnitude of the four-momentum is

$$p^2 = p \cdot p = g_{\mu\nu} p^\mu p^\nu = p_\nu p^\nu = E^2 - |\vec{p}|^2 = E^2 - p_x^2 - p_y^2 - p_z^2, \tag{2.2}$$

where $g_{\mu\nu}$ is the (covariant form of the) metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.3}$$

We have used *Einstein summation convention* in the expression for p^2; that is, repeated indices are implicitly summed over. According to the theory of relativity, the squared magnitude of the four-momentum of a physical particle is equal to its invariant rest-mass.

The contravariant form of the metric, $g^{\mu\nu}$, is defined by

$$g^{\mu\alpha} g_{\mu\beta} = \delta^{\alpha}_{\beta} = g_{\mu\beta} g^{\mu\alpha} \qquad g^{\mu\nu} g_{\mu\nu} = d, \tag{2.4}$$

where δ^{α}_{β} is the Kronecker delta symbol and $d = 4$ is the dimension of spacetime. In the present case (flat spacetime with Cartesian coordinates), it happens that $g^{\mu\alpha}$ is identical to $g_{\mu\alpha}$; this is not true in general. The metric tensor has the property of raising and lowering indices:

$$g^{\mu\nu} p_{\mu} = p^{\nu} \qquad g_{\mu\nu} p^{\mu} = p_{\nu}. \tag{2.5}$$

Along with the metric, we can define the Levi-Civita tensor density in four dimensions:

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \{\mu,\nu,\alpha,\beta\} \text{ is an even permutation of } \{0,1,2,3\}, \\ -1 & \text{odd permutation,} \\ 0 & \text{if two or more indices are repeated.} \end{cases} \tag{2.6}$$

It follows from its construction that

$$\begin{aligned} \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}{}^{\nu'\alpha'\beta'} =& g^{\nu\alpha'}g^{\alpha\nu'}g^{\beta\beta'} + g^{\nu\nu'}g^{\alpha\beta'}g^{\beta\alpha'} + g^{\nu\beta'}g^{\alpha\alpha'}g^{\beta\nu'} \\ & - g^{\nu\nu'}g^{\alpha\alpha'}g^{\beta\beta'} - g^{\nu\beta'}g^{\alpha\nu'}g^{\beta\alpha'} - g^{\nu\alpha'}g^{\alpha\beta'}g^{\beta\nu'} \ . \end{aligned} \tag{2.7}$$

By further contracting (2.7), we determine

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}{}^{\alpha'\beta'} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu}{}^{\nu'\alpha'\beta'}g_{\nu'\nu} = 2(g^{\alpha\alpha'}g^{\beta\beta'} - g^{\alpha\beta'}g^{\beta\alpha'}) \ , \tag{2.8}$$

which can itself be contracted.

The fermionic particles, which constitute matter as we know it, are described by spinors. Algebraic computation with spinors requires the use of Dirac matrices. These matrices satisfy the axioms of a Clifford Algebra, and a particularly convenient representation of them is quoted below [2]:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} , \quad i \in \{1,2,3\} \tag{2.9}$$

where each element in the matrices is actually a 2×2 block with I equal to the identity matrix and σ_i the ith Pauli matrix. The four Dirac matrices obey the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.10}$$

which is the condition that they form a Clifford Algebra. In addition, one can construct another object, unique up to representation, which we denote γ^5,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.11}$$

which also anticommutes with each of the original gamma matrices:

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5. \tag{2.12}$$

Since the Dirac matrices assemble to form a matrix-valued four-vector, it is natural that they be contracted with other four-vectors. The shorthand notation for this is "slash," whereby

$$p_\mu\gamma^\mu = p^\mu\gamma_\mu = \not{p}. \tag{2.13}$$

An incoming particle is designated by the Dirac spinor, $U^{(s)}(p)$, which can also describe an outgoing antiparticle. The adjoint Dirac spinor, $\bar{U}^{(s)}(p)$, designates an outgoing particle or an incoming antiparticle,

$$\bar{U}^{(s)}(p) = U^{\dagger(s)}(p)\gamma^0. \tag{2.14}$$

The spinor and the adjoint spinor are a 4×1 column vector and a 1×4 row vector, respectively. The parameter (s) in the superscript denotes the spin which may be $(\pm\frac{1}{2})$

or up/down; the parameter p denotes the momentum. These external particles must be on-shell, i.e., satisfy the Dirac equation,

$$(\not{p} - m)U(p) = 0, \tag{2.15}$$

and its adjoint. The physical spinors obey a completeness-like relation of the form

$$\sum_s U^{(s)}(p)\bar{U}^{(s)}(p) = \not{p} + m. \tag{2.16}$$

In formulae like (2.10 and 2.16) the factors which are not manifestly matrix-valued are understood to be coefficients of a 4×4 identity matrix.

2.2. Lagrangian and Feynman rules

We begin with the Lagrangian which has been proposed by a number of authors as a simple gauge–invariant model of an antisymmetric tensor field in interaction with fundamental, point-like fermions [18, 19].

$$\mathcal{L} = -\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \overline{\psi}\left(i\,\not{\partial} - \frac{g}{9M}\sigma_{\mu\nu\lambda}H^{\mu\nu\lambda} - m\right)\psi, \tag{2.17}$$

where $-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}$ is the free Lagrangian density, or a kinetic term, for the antisymmetric tensor field. The field strength, $H_{\mu\nu\lambda}$, associated with the antisymmetric tensor,

$B_{\alpha\beta} \equiv -B_{\beta\alpha}$, is defined as

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}. \tag{2.18}$$

In the language of differential forms, $H_{\mu\nu\lambda}$ is the exterior derivative of the 2-form $B_{\mu\nu}$. In tensorial terms, $H_{\mu\nu\lambda}$ is equal to three times the generalized curl of $B_{\mu\nu}$. The free ("non-interacting") Lagrangian density for the fermion represented by the spinor, ψ, and its adjoint, $\overline{\psi}$, is $\overline{\psi}(i \not\partial - m)\psi$. The term which is trilinear in the fields, $\overline{\psi}(-\frac{g}{9M}\sigma_{\mu\nu\lambda}H^{\mu\nu\lambda})\psi$, describes the coupling of the fermionic field to the antisymmetric tensor field that we choose to consider in our model. The Dirac algebra factor needed to contract the spinor indices mirrors the antisymmetry of the field strength and is the completely antisymmetric product of $\gamma_\alpha\gamma_\beta\gamma_\gamma$:

$$\sigma_{\alpha\beta\gamma} = i\epsilon_{\alpha\beta\gamma\mu}\gamma_5\gamma^\mu, \tag{2.19}$$

following from the identity

$$\gamma^\alpha\gamma^\beta\gamma^\gamma \equiv \gamma^\alpha g^{\beta\gamma} - \gamma^\beta g^{\alpha\gamma} + \gamma^\gamma g^{\alpha\beta} + i\epsilon^{\alpha\beta\gamma\mu}\gamma^5\gamma_\mu. \tag{2.20}$$

The precise form of the coupling is motivated by considerations of gauge invariance and is not quite minimal, nor is it unique. Other aspects of this model, in conjuction with Quantum Electrodynamics (QED), are studied in [18, 19].

We derive the Feynman rules from the Lagrangian. They will tell us how the antisymmetric tensor interacts with fermions. There is a standard method for extracting the rules from a given Lagrangian [1, 2, 3, 4, 5, 6, 7, 8].

The Feynman rule describing the propagator of the antisymmetric tensor is complicated by the issue of gauge invariance. According to the standard construction, the propagator is the inverse of the quadratic part of the Lagrangian. In fact, the quadratic piece, as it stands, is not invertible on account of the gauge invariance which guided us to the specific form we have chosen. To proceed, we must first add a gauge breaking piece (which introduces an irrelevant gauge-fixing parameter). Once the propagator is obtained, we choose a value for the gauge-fixing parameter which casts the propagator into a simple and convenient form. For details, see [19]. The form that we shall use is

$$G^{\nu\alpha\lambda\beta} = -\frac{g^{\nu\alpha}g^{\lambda\beta} - g^{\nu\beta}g^{\lambda\alpha}}{q^2} \ , \tag{2.21}$$

where q_ν is the four-momentum carried by the antisymmetric tensor particle being propagated.

There is another Feynman rule for the fermion-antisymmetric-tensor vertex. This rule is extracted directly from the interaction Lagrangian (2.17),

$$\mathcal{L}_{\mathsf{int}} = -\frac{g}{9M}\overline{\psi}\ \sigma_{\mu\nu\lambda}H^{\mu\nu\lambda}\ \psi, \tag{2.22}$$

and re-expressed in terms of the tensor field $B_{\alpha\beta}$,

$$\mathcal{L}_{\mathsf{int}} \equiv -\frac{g}{9M}\overline{\psi}\ \Lambda^{\alpha\beta}\ \psi\ B_{\alpha\beta}. \tag{2.23}$$

Substituting the definitions of $\sigma_{\mu\nu\lambda}$ and $H^{\mu\nu\lambda}$ (2.18 and 2.19), and Fourier transforming, we arrive at the vertex rule that couples the antisymmetric tensor and fermions:

$$-\frac{g}{9M}\Lambda^{\alpha\beta} = -\frac{g}{9M}\epsilon_{\mu\nu\lambda\sigma}\gamma_5\gamma^\sigma \left(g^{\nu\alpha}g^{\lambda\beta}q^\mu + g^{\lambda\alpha}g^{\mu\beta}q^\nu + g^{\mu\alpha}g^{\nu\beta}q^\lambda\right). \tag{2.24}$$

There is an overall energy–momentum conserving delta-function implicit in the expression for the vertex. We have chosen not to write it explicitly in order to focus more clearly on the tensorial and spinorial aspects of the vertex.

Figure 2.1 summarizes the Feynman rules for the vertex and the propagator. Diagrammatically, the vertex is represented by the joint where two solid lines meet the dashed line. The dashed line itself is the propagator.

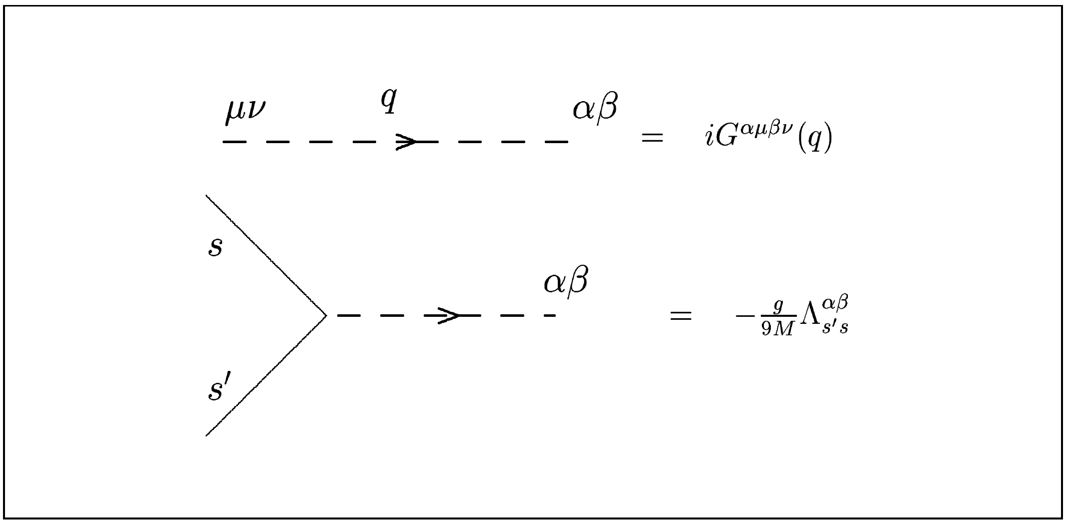


Figure 2.1. The minimal set of Feynman rules needed for the computations in Chapter 3.

CHAPTER 3

TREE-LEVEL SCATTERING

Here we examine two distinct scattering processes of external fermions via anti-symmetric tensor exchange. Sections 3.1 and 3.2 describe the scattering of two muons (identical fermions of non-negligible mass). The subsequent section examines the scenario of one electron and one muon scattering (distinct fermions). Each calculation will follow these steps: Feynman diagram(s), probability amplitude, differential cross section, and total cross section. In the identical fermion case, a divergence occurs in the total cross section. We will comment on the likely solution to this problem in the conclusion.

3.1. Muon-muon probability amplitude

Consider the Feynman diagrams that describe the interaction. The outgoing muons, because they are identical, cannot be distinguished by any observer. Thus, the two diagrams in Figure 3.1 are necessary to account for the two possibilities. From reading the diagrams, we can write down the $\mathcal{M}$-matrix:

$$\begin{aligned} i\mathcal{M} =& \bar{U}^{(s')}(p')\Big(- \frac{g}{9M}\Lambda^{\alpha\beta}_{s's}(q)\Big)U^{(s)}(p)\Big(iG_{\alpha\mu\beta\nu}(q)\Big)\bar{U}^{(r')}(k')\Big(- \frac{g}{9M}\Lambda^{\mu\nu}_{r'r}(q)\Big)U^{(r)}(k) \\ & - (k' \leftrightarrow p', r' \leftrightarrow s', q \rightarrow \tilde{q}). \end{aligned} \tag{3.1}$$

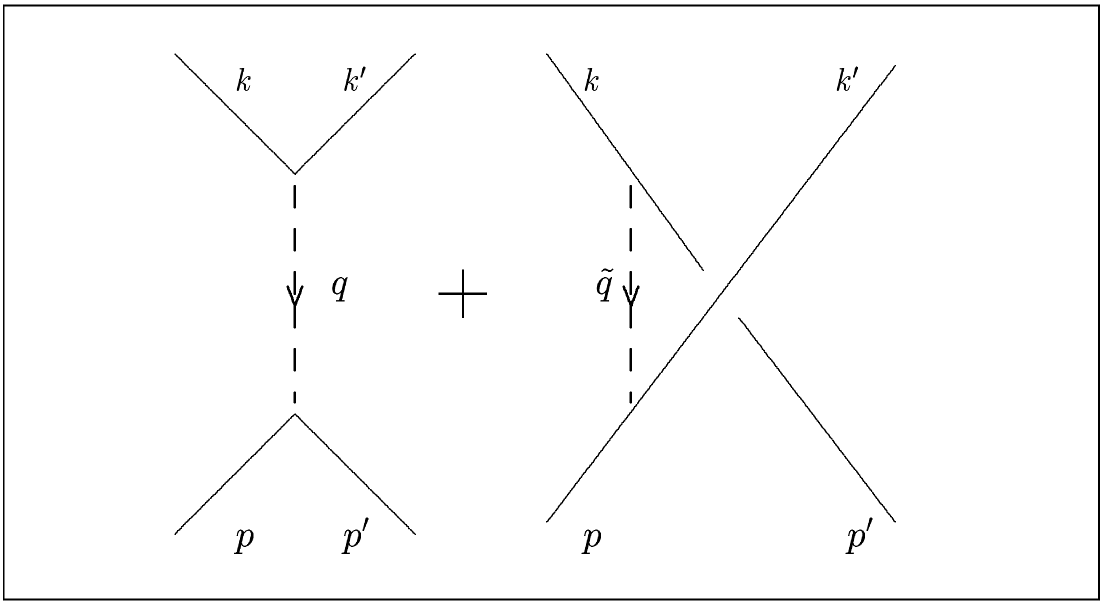


Figure 3.1. The Feynman diagrams for the scattering of two identical fermions via the exchange of an antisymmetric tensor field.

According to Wick's Theorem [1, 2, 3, 4, 5, 6, 7, 8], the exchange of fermions in the final state of the second diagram leads to a relative minus sign in front of the second term. The vertex, $\Lambda^{\alpha\beta}_{s's}(q)$, has a tensor part with $\alpha\beta$ indices and a spinor part with spin indices ss'. It can be rewritten to separate the two:

$$\Lambda^{\alpha\beta}_{s's}(q) = \Lambda^{\alpha\beta}_{\delta}(q), (\gamma_5\gamma^{\delta})_{s's}, \tag{3.2}$$

where

$$\Lambda^{\alpha\beta}_{\delta}(q) = \epsilon_{abc\delta}(g^{b\alpha}g^{c\beta}q^a + g^{c\alpha}g^{a\beta}q^b + g^{a\alpha}g^{b\beta}q^c). \tag{3.3}$$

We recall that the form of our gauge-fixed propagator for the antisymmetric field (2.21) is equivalent to

$$G_{\alpha\mu\beta\nu}(q) = -\frac{1}{q^2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

in covariant form. Equipped with our knowledge of $\epsilon_{abc\delta}$ and $g^{\mu\nu}$ listed in Chapter 2, in particular equations (2.4), (2.7), and (2.8), we arrive at

$$\Lambda_\lambda^{\alpha\beta} G_{\alpha\mu\beta\nu} \Lambda_\delta^{\mu\nu} = 36\left(\frac{q_\lambda q_\delta}{q^2} - g_{\delta\lambda}\right) . \tag{3.4}$$

Substituting (3.4) into (3.1) and applying the Dirac equation to the external particle states, the $\mathcal{M}$-matrix becomes

$$\begin{aligned}\mathcal{M} = 36(\frac{g}{9M})^2\Big(&- \frac{4m^2}{q^2}\bar{U}(p')\gamma_5 U(p)\bar{U}(k')\gamma_5 U(k) - \bar{U}(p')\gamma_5\gamma^\delta U(p)\bar{U}(k')\gamma_5\gamma_\delta U(k) \\ &+ \frac{4m^2}{\tilde{q}^2}\bar{U}(k')\gamma_5 U(p)\bar{U}(p')\gamma_5 U(k) + \bar{U}(k')\gamma_5\gamma^\delta U(p)\bar{U}(p')\gamma_5\gamma_\delta U(k)\Big).\end{aligned} \tag{3.5}$$

Constructing the probability amplitude from $\mathcal{M}$ is similar to finding the probability from the wave function in quantum mechanics. That is, the probability is the squared magnitude of the $\mathcal{M}$-matrix. We note, however, that there are different, distinct spin states available to the fermions in the initial and final states. Since spins can, in fact, be uniquely specified and/or measured, their contributions to the scattering probability add without interference. Hence, we must sum the probabilities associated with the different combinations of spins. In practice, it is usually the case in situations involving the scattering of two fermions, as considered here, that the initial beams have no net polarization. That is, we expect half of the incoming particles to be spin up and the other half spin down. In this case, we simply "average" over the initial polarizations, assuming that each occurs in precisely half of the particles, then sum over the possible final spin states. Thus, the total probability is given by

$$(\frac{1}{2}\sum_{s})\sum_{s'}(\frac{1}{2}\sum_{r})\sum_{r'}|\mathcal{M}|^2 = (\frac{1}{2}\sum_{s})\sum_{s'}(\frac{1}{2}\sum_{r})\sum_{r'}\mathcal{M}^\dagger\mathcal{M} \ , \tag{3.6}$$

where $\mathcal{M}^\dagger$ is the complex conjugate transpose of $\mathcal{M}$. Performing the sums and averaging, we make use of identity (2.16) quoted in Chapter 2 and obtain

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 324(\frac{g}{9M})^4 \begin{bmatrix} A & + & B & + & C & + & D \\ +E & + & F & + & G & + & H \\ +I & + & J & + & K & + & L \\ +M & + & N & + & O & + & P \end{bmatrix}, \tag{3.7}$$

where

$$A = \frac{16m^4}{q^4}\mathrm{Tr}[(\not p + m)\gamma_5(\not p\,' + m)\gamma_5]\mathrm{Tr}[(\not k + m)\gamma_5(\not k\,' + m)\gamma_5],$$

$$B = \frac{4m^2}{q^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not p\,' + m)\gamma_5\gamma^\delta]\mathrm{Tr}[(\not k + m)\gamma_5(\not k\,' + m)\gamma_5\gamma_\delta],$$

$$C = -\frac{16m^4}{q^2\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not p\,' + m)\gamma_5(\not k + m)\gamma_5(\not k\,' + m)\gamma_5],$$

$$D = -\frac{4m^2}{q^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not p\,' + m)\gamma_5\gamma_\delta(\not k + m)\gamma_5(\not k\,' + m)\gamma_5\gamma^\delta],$$

$$E = \frac{4m^2}{q^2}\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not p\,' + m)\gamma_5]\mathrm{Tr}[(\not k + m)\gamma_\lambda\gamma_5(\not k\,' + m)\gamma_5],$$

$$F = \mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not p\,' + m)\gamma_5\gamma^\delta]\mathrm{Tr}[(\not k + m)\gamma_\lambda\gamma_5(\not k\,' + m)\gamma_5\gamma_\delta],$$

$$G = -\frac{4m^2}{\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not p\,' + m)\gamma_5(\not k + m)\gamma_\lambda\gamma_5(\not k\,' + m)\gamma_5],$$

$$H = -\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not p\,' + m)\gamma_5\gamma_\delta(\not k + m)\gamma_\lambda\gamma_5(\not k\,' + m)\gamma_5\gamma^\delta],$$

$$I = -\frac{16m^4}{q^2\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not k\,' + m)\gamma_5(\not k + m)\gamma_5(\not p\,' + m)\gamma_5],$$

$$J = -\frac{4m^2}{\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not k\,' + m)\gamma_5\gamma_\delta(\not k + m)\gamma_5(\not p\,' + m)\gamma_5\gamma^\delta],$$

$$K = \frac{16m^4}{\tilde{q}^4}\mathrm{Tr}[(\not p + m)\gamma_5(\not k\,' + m)\gamma_5]\mathrm{Tr}[(\not k + m)\gamma_5(\not p\,' + m)\gamma_5],$$

$$L = \frac{4m^2}{\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma_5(\not k\,' + m)\gamma_5\gamma^\delta]\mathrm{Tr}[(\not k + m)\gamma_5(\not p\,' + m)\gamma_5\gamma_\delta],$$

$$M = -\frac{4m^2}{q^2}\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not k\,' + m)\gamma_5(\not k + m)\gamma_\lambda\gamma_5(\not p\,' + m)\gamma_5],$$

$$N = -\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not k\,' + m)\gamma_5\gamma_\delta(\not k + m)\gamma_\lambda\gamma_5(\not p\,' + m)\gamma_5\gamma^\delta],$$

$$O = \frac{4m^2}{\tilde{q}^2}\mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not k\,' + m)\gamma_5]\mathrm{Tr}[(\not k + m)\gamma_\lambda\gamma_5(\not p\,' + m)\gamma_5],$$

and

$$P = \mathrm{Tr}[(\not p + m)\gamma^\lambda\gamma_5(\not k\,' + m)\gamma_5\gamma^\delta]\mathrm{Tr}[(\not k + m)\gamma_\lambda\gamma_5(\not p\,' + m)\gamma_5\gamma_\delta].$$

We use the following properties of γ traces,

$$\begin{aligned} &\mathrm{Tr}(\not p \not p\,') = 4p\cdot p'\ , \\ &\mathrm{Tr}(\not a \not b \not c \not d) = 4(a\cdot b c\cdot d - a\cdot c b\cdot d + a\cdot d b\cdot c)\ , \\ &\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\ , \end{aligned} \tag{3.8}$$

and that the trace of odd number of γ's is identically zero, to simplify the expression for the probability:

$$
\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 324(\frac{g}{9M})^4\Big[&\frac{256m^4}{q^4}\Big(m^4 - m^2 p\cdot p' - m^2 k\cdot k' + p\cdot p' k\cdot k'\Big)\\
+ \frac{64m^2}{q^2}\Big(&-m^2(p\cdot k' + p'\cdot k - p\cdot k - p'\cdot k' - 2p\cdot p' - 2k\cdot k')\\
&+ 2m^2(p\cdot k - p\cdot k' - p'\cdot k + p'\cdot k') - 2m^4\\
&- p\cdot k' p'\cdot k - p\cdot p' k\cdot k'\Big)\\
- \frac{128m^4}{q^2\tilde{q}^2}\Big(&m^4 + m^2(p\cdot k - p\cdot p' - p\cdot k' - p'\cdot k - k\cdot k' + p'\cdot k')\\
&+ p\cdot k' p'\cdot k - p\cdot k p'\cdot k' + p\cdot p' k\cdot k'\Big)\\
+ 32\Big(&m^2(2p\cdot p' + 2k\cdot k' + 2p\cdot k' + 2p'\cdot k - p\cdot k - p'\cdot k')\\
&+ 4p\cdot k p'\cdot k' + p\cdot k' p'\cdot k + p\cdot p' k\cdot k' + 6m^4\Big)\\
+ \frac{64m^2}{\tilde{q}^2}\Big(&m^2(3p\cdot k - 3p\cdot p' - 3k\cdot k' + 3p'\cdot k' + 2p\cdot k' + 2p'\cdot k)\\
&- 2m^4 - p\cdot p' k\cdot k' - p\cdot k' p'\cdot k\Big)\\
+ \frac{256m^4}{\tilde{q}^4}\Big(&m^4 - m^2 p\cdot k' - m^2 p'\cdot k + p\cdot k' p'\cdot k\Big)\Big]\ .
\end{aligned}
\tag{3.9}
$$

This result characterizes the dynamics of antisymmetric-tensor-mediated interaction between identical massive fermions. Kinematical considerations will enter in the next section when we choose a frame for continuing the analysis.

3.2. Muon-muon cross section

We must eventually choose a kinematical frame in which to work. Simplicity dictates that we use the center of momentum frame since we have identical incoming particles. The virtues of this choice include its respect of the symmetry of the initial state, the simplifications that come from having the total initial and final three-momenta equal to zero, and the fact that it is coincident with the "detector frame" in a collider setting.

If we picture the interaction occurring in a collider with the two incoming particles approaching each other head on, we can imagine that they are on the same axis. Furthermore, we can adopt spherical polar coordinates and identify this axis as the z axis, with the origin located at the point where the collision occurs.

From Figure 3.2, we see that the energy-momentum four-vectors are

$$\begin{aligned} p &= (E, \alpha\hat{z}) & k &= (E, -\alpha\hat{z}) \\ p' &= (E, -\vec{\beta}) & k' &= (E, \vec{\beta}). \end{aligned} \tag{3.10}$$

The components of these vectors are constrained by the physical requirements that $E^2 - \alpha^2 = m^2$, and since the collision is elastic, $|\vec{\beta}|^2 = \alpha^2$. The dot products appearing in the probability amplitude can be explicitly expressed in terms of the energy of the incident beams, the rest mass of the fermions, and the scattering angle:

$$\begin{aligned} p \cdot k &= 2E^2 - m^2 & p \cdot p' &= E^2(1+\cos\theta) - m^2\cos\theta \\ p \cdot k' &= E^2(1-\cos\theta) + m^2\cos\theta & p' \cdot k &= E^2(1-\cos\theta) + m^2\cos\theta \\ p' \cdot k' &= 2E^2 - m^2 & k \cdot k' &= E^2(1+\cos\theta) - m^2\cos\theta \end{aligned} \tag{3.11}$$

Substituting equation (3.11) into the probability amplitude and making cancellations to eliminate the q^2, $\tilde{q}^2$, q^4, $\tilde{q}^4$, and $q^2\tilde{q}^2$ factors which appear in various denominators,

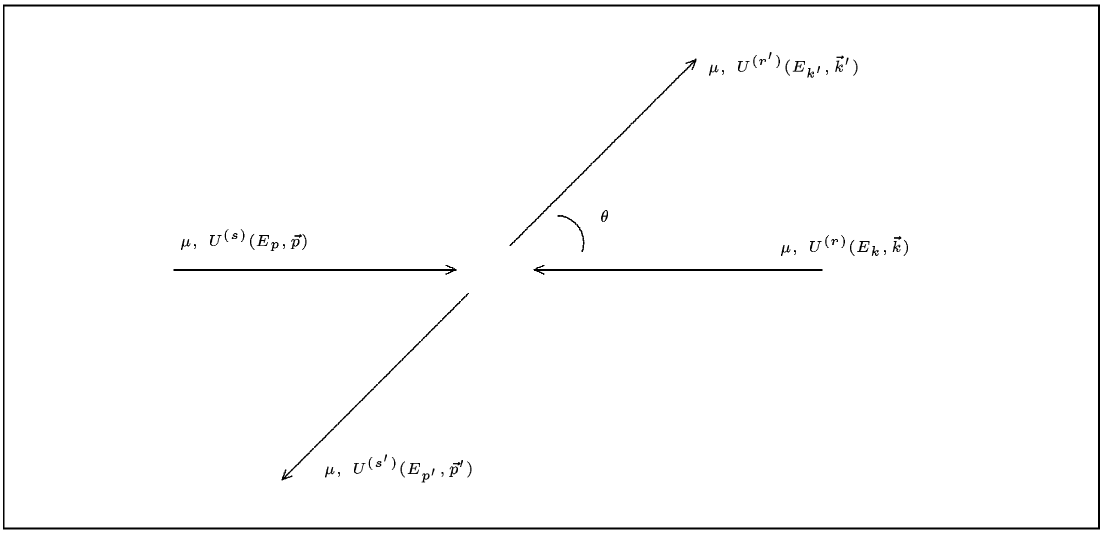


Figure 3.2. The kinematics of the muon-muon collision.

we arrive at

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 324\cdot 64(\frac{g}{9M})^4\Big[m^4 + m^4\cos^2\theta - 4m^2E^2 - 2m^2E^2\cos^2\theta + 4m^2E^2\frac{1}{\tan^2\theta} + 9E^4 + E^4\cos^2\theta\Big]\ . \tag{3.12}$$

The *differential cross section* tells us how likely it is that the outgoing particles will be observed in a given solid angle. The differential cross section is a function of polar angle θ, ranging from 0 to π, and not of the azimuthal angle ϕ, ranging from 0 to 2π, on account of the axial symmetry of the system. To make the transition from probability amplitude to the differential cross section, we must introduce the *Lorentz invariant phase space factor* [7]. Doing so, we finally arrive at the expression for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2}\frac{E}{16\pi^2 E_{cm}}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\ . \tag{3.13}$$

In this equation, E_{cm} is the total energy in the center of mass frame, i.e., $E_{cm} = 2E$. Equation (3.13) provides a measure of the likelihood for a scattering process to result in the final state fermions leaving the interaction region in the solid angle $d\Omega$ centered on θ and ϕ. However, the axial symmetry of the collision geometry precludes ϕ dependence.

We can attempt to integrate equation (3.13) to obtain the total cross section for the process of single antisymmetric particle exchange. We obtain

$$\int \frac{d\sigma}{d\Omega} d\Omega = \frac{g^4}{\pi M^4}\left[\frac{64}{243}\frac{m^4}{E_{cm}^2} + m^2\left(\frac{8}{81}\int_0^{\pi}\frac{\sin\theta}{\tan^2\theta}d\theta - \frac{56}{243}\right) + \frac{28}{243}E_{cm}^2\right]. \tag{3.14}$$

Take note that we have a divergence in one term. It poses a problem as it would suggest that the cross section is infinite. Upon close examination, we notice that the integral diverges only when θ takes on values of 0 and π, that is, along the incident beams axis. Experimentally, we cannot put detectors there anyway. In the next chapter, we will propose a possible solution to this problem.

3.3. Electron-muon probability amplitude and cross section

The calculations in this case, with distinctly different masses for the incoming particles are much like those in the previous sections. For brevity we will only make note of significant differences. Once again we begin with a Feynman diagram that describes the interaction as displayed in Figure 3.3.

There is only one diagram–and thus no interference–because the particles which scatter are distinguishable. The $\mathcal{M}$-matrix from the diagram becomes, after the spin

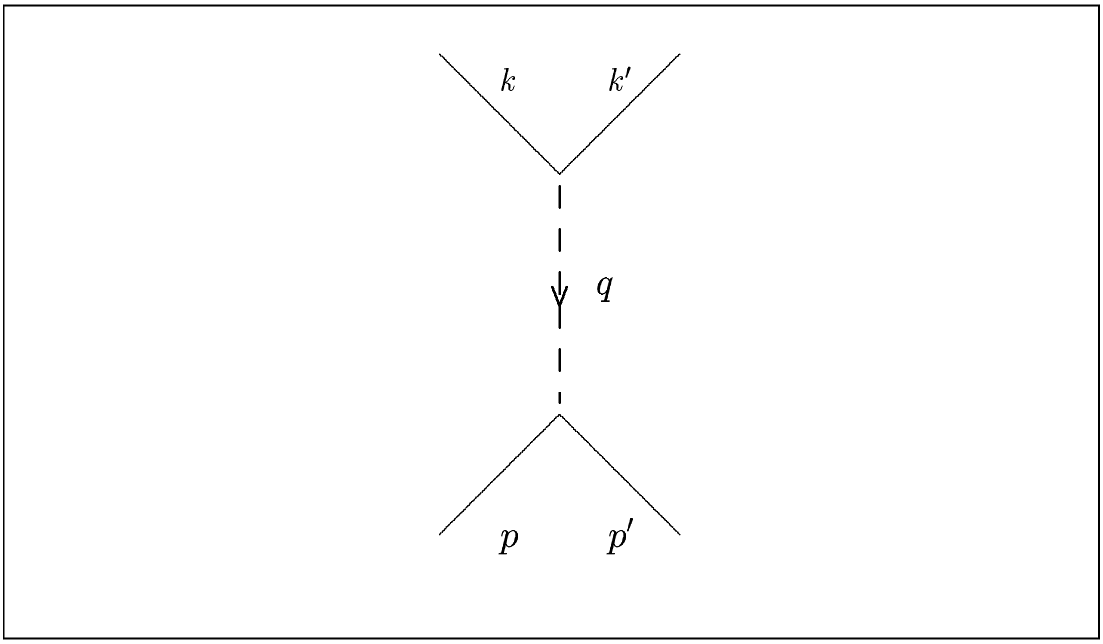


Figure 3.3. The Feynman diagram for the scattering of two distinct fermions via the exchange of an antisymmetric tensor field.

part is separated and the Dirac equation applied,

$$\mathcal{M} = 36(\frac{g}{9M})^2\Big(\frac{-4m\tilde{m}}{q^2}\bar{U}(p')\gamma_5 U(p)\bar{U}(k')\gamma_5 U(k) - \bar{U}(p')\gamma_5\gamma^\delta U(p)\bar{U}(k')\gamma_5\gamma_\delta U(k)\Big). \tag{3.15}$$

Constructing the probability amplitude from $\mathcal{M}$ by taking the squared magnitude, summing over final spin states, and averaging over the initial spins gives

$$\begin{aligned}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 324(\frac{g}{4M})^4\Big[&\frac{16m^2\tilde{m}^2}{q^4}\mathrm{Tr}[(\not{p}+m)(m-\not{p}\,')]\mathrm{Tr}[(\not{k}+\tilde{m})(\tilde{m}-\not{k}\,')] \\ &+\frac{4m\tilde{m}}{q^2}\mathrm{Tr}[(\not{p}+m)(m-\not{p}\,')\gamma^\delta]\mathrm{Tr}[(\not{k}+\tilde{m})(\tilde{m}-\not{k}\,')\gamma_\delta] \\ &+\frac{4m\tilde{m}}{q^2}\mathrm{Tr}[(\not{p}+m)\gamma^\lambda(m-\not{p}\,')]\mathrm{Tr}[(\not{k}+\tilde{m})\gamma_\lambda(\tilde{m}-\not{k}\,')] \\ &+\mathrm{Tr}[(\not{p}+m)\gamma^\lambda(m-\not{p}\,')\gamma^\delta]\mathrm{Tr}[(\not{k}+\tilde{m})\gamma_\lambda(\tilde{m}-\not{k}\,')\gamma_\delta]\Big]\,.\end{aligned} \tag{3.16}$$

Working out the traces by application of the trace identities yields

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = 10368(\frac{g}{4M})^4\Big[\frac{8m^2\tilde{m}^2}{q^4}(m^2\tilde{m}^2 - m^2 k\cdot k' - \tilde{m}^2 p\cdot p' + p\cdot p' k\cdot k') + \frac{4m^2\tilde{m}^2}{q^4}(p\cdot k - p\cdot k' - p'\cdot k + p'\cdot k') + (p\cdot k p'\cdot k' + p\cdot k' p'\cdot k + \tilde{m}^2 p\cdot p' + m^2 k\cdot k' + 2m^2\tilde{m}^2)\Big] \quad (3.17)$$

for the invariant scattering amplitude.

For the cross-section calculation, despite the electron and muon having different masses, we choose the center of momentum frame in which the total three-momentum, initial and final, vanishes. Although the muon is ~ 207 times more massive than the electron, we can always choose a frame in which to observe the collision where their momenta are equal in magnitude. Of course, it requires fine tuning to make this frame correspond with the detector frame, so the final results for the cross section may have to be boosted to compare with experiment.

The energy-momentum four-vectors may be written as

$$p = (E_1, \alpha\hat{z}) \qquad k = (E_2, -\alpha\hat{z})$$
$$p' = (E_3, -\vec{\beta}) \qquad k' = (E_4, \vec{\beta})\ . \quad (3.18)$$

Working out the non-trivial dot products of the various momenta with reference to

Figure 3.4 yields

$$
\begin{array}{ll}
p \cdot k = E_1 E_2 + \alpha^2 & p \cdot p' = E_1 E_3 + \alpha|\vec{\beta}| \cos\theta \\
p \cdot k' = E_1 E_4 - \alpha|\vec{\beta}| \cos\theta & p' \cdot k = E_2 E_3 - \alpha|\beta| \cos\theta \\
p' \cdot k' = E_3 E_4 + |\vec{\beta}|^2 & k \cdot k' = E_2 E_4 + \alpha|\beta| \cos\theta \ .
\end{array}
\tag{3.19}
$$

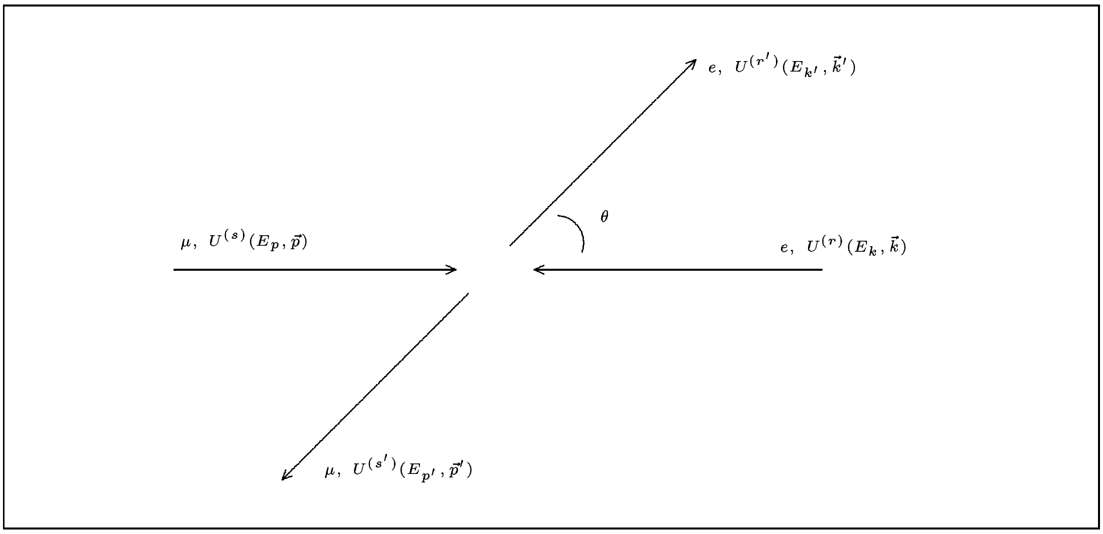


Figure 3.4. The kinematics of the electron-muon collision.

Substituting equation (3.19) into the probability amplitude, and again cancelling the q^2 and q^4 factors in the denominator, we arrive at

$$
\begin{aligned}
\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = 10368(\frac{g}{4M})^4 \Big[& 2E_1E_2E_3E_4 + E_1E_2|\beta|^2 + E_3E_4\alpha^2 \\
& + \alpha^2|\vec{\beta}|^2(1+\cos^2\theta) + \tilde{m}^2 E_1E_3 + m^2 E_2E_4 \\
& + \alpha|\vec{\beta}|\cos\theta(m^2 + \tilde{m}^2 - E_1E_4 - E_2E_3)\Big]
\end{aligned}
\tag{3.20}
$$

as our result in this more complicated case. The Lorentz invariant phase space factor is slightly different:

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{\beta}|}{2 \cdot 16\pi^2 (E_1 + E_2)^3} \frac{1}{4} \sum_{spins} |\mathcal{M}|^2 \tag{3.21}$$

on account of the loss of symmetry in the initial and final states. Formally, we integrate to obtain the total cross section:

$$\begin{aligned} \int \frac{d\sigma}{d\Omega} d\Omega = \frac{81 |\vec{\beta}| g^4}{16\pi (E_1 + E_2)^3 M^4} \Big[& 2E_1 E_2 E_3 E_4 + E_1 E_2 |\vec{\beta}|^2 + E_3 E_4 \alpha^2 \\ & + \frac{4}{3} \alpha^2 |\vec{\beta}|^2 + \tilde{m}^2 E_1 E_3 + m^2 E_2 E_4 \Big] \ . \end{aligned} \tag{3.22}$$

Note that the energies of the particles which appear in this formula are constrained in a number of ways. First and foremost is by conservation of energy since the one-particle exchange process is intrinsically elastic; therefore,

$$E_1 + E_2 \equiv E_3 + E_4 \ . \tag{3.23}$$

Second, the incident and outgoing particles are on mass-shell and, thus, obey

$$\begin{aligned} E_1^2 &= m^2 + \alpha^2 & \qquad E_3^2 &= m^2 + |\vec{\beta}|^2 \\ E_2^2 &= \tilde{m}^2 + \alpha^2 & \qquad E_4^2 &= \tilde{m}^2 + |\vec{\beta}|^2 \end{aligned} \tag{3.24}$$

in order to describe the physical particles which are observed far from the interaction region. The presence of these constraints complicates the analysis and makes it difficult to extract the shape of the differential cross section as well as the magnitude of the total cross section.

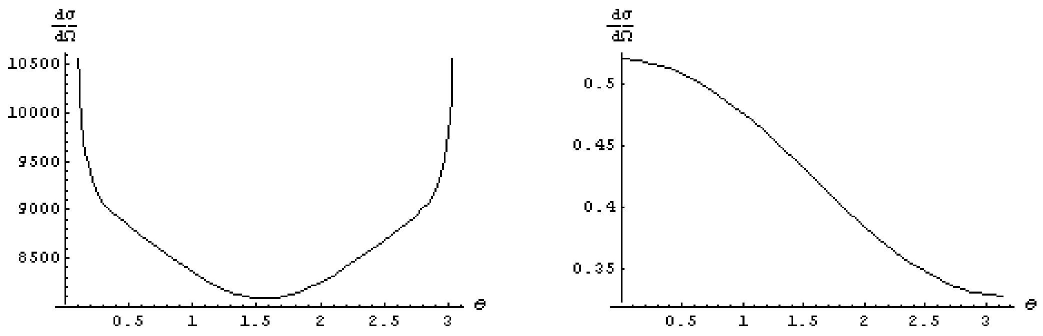
CHAPTER 4

DISCUSSION AND CONCLUSIONS

We begin our analysis with a discussion of the angular dependence of the differential cross sections. In the $\mu{+}\mu \rightarrow \mu{+}\mu$ differential cross section, equations (3.12) and (3.13), the polar angle θ appears in terms of the form $\cos^2\theta$ and $\frac{1}{\tan^2\theta}$ which are symmetric under $\theta \rightarrow \pi - \theta$. This symmetry indicates that forward scattering and backward scattering of particles are equally probable. Figure 4.1(a) shows the symmetry of the $\mu{+}\mu \rightarrow \mu{+}\mu$ differential cross section as a function of the polar angle. We chose the energy for the incoming and outgoing muons to be fifteen times its mass, a representative value to show the general behavior.

A plot of the differential cross section for $e + \mu \rightarrow e + \mu$ is shown in Figure 4.1(b). Here we chose the energy for the incoming and outgoing electrons to be twice its mass. Notice the asymmetry; it tells us that low-angle scattering is more likely. This asymmetry is an indication of a preference to backward scattering of the electron.

In the total cross section for identical fermions, equation (3.14), our computation shows a divergence. A similar problem arises in QED, and there is a well-established method for dealing with such *Infra-red divergences* [1, 2, 4, 6]. This method is often called Infra-red renormalization (regularization) or the *Bloch-Nordsieck* approach. The essence of the idea is that there are *inelastic* processes corresponding to the emission of very-low-energy massless gauge particles (photons in QED) which cannot be



(a) (b)

Figure 4.1. Representative plots demonstrating the polar angle dependence of the differential cross sections for (a) $\mu + \mu \rightarrow \mu + \mu$ and (b) $e + \mu \rightarrow e + \mu$ via one-particle antisymmetric tensor exchange.

distinguished experimentally from the elastic scattering process of one-photon exchange in the Born Approximation. Despite the fact that each inelastic process has a finite cross section, the infinite ensemble of possibilities produces a divergent result. Careful analysis, first conducted by Bloch and Nordsieck long ago [4], shows that the consideration of these two processes (elastic and inelastic ensemble) together precisely cancels the divergence.

We strongly suspect that the analogous procedure will work here to cancel the divergent term in the $\mu + \mu \rightarrow \mu + \mu$ total cross section. Verifying this conjecture is a topic for future work.

It is curious to note the absence of a divergence in the $e + \mu \rightarrow e + \mu$ cross section. We initially expected a similar divergent term to appear. However, the actual divergence emerged from the two cross terms (interference) in the probability amplitude for $\mu + \mu \rightarrow$

$\mu + \mu$. The $e + \mu \to e + \mu$ process is completely described by a single Feynman diagram, thus no interference is possible.

Setting the divergence concern aside, we now turn our attention to another aspect of the functional forms of the total cross sections. We notice that both total cross sections grow as functions of incident beam energy squared. Cross sections typically decrease as energy increases. At first glance, this observation appears troublesome as it would imply that the cross section grows without bound as the energy increases. Recall that the interaction between the fermions and the antisymmetric tensor field proceeds via the field strength, $H_{\mu\nu\lambda}$, and thus explicitly involves derivatives. With two vertices, there are two derivatives, each contributing an additional factor of energy-momentum to the amplitude which is then squared to form the cross section. These factors explain the origin of this particular dependence on energy.

Furthermore, our analysis cannot neglect the quantitative effect of the coupling constant(s) which enter into our formulae for the cross sections. The factor, $(\frac{g}{M})^4$, ensures that the cross sections have the correct overall dimension and can suppress the growth with energy. An upper bound for the value of the coupling has been obtained from an independent analysis of the anomalous magnetic moment of the muon [19]. The result is

$$\frac{g^2}{M^2} \leq 2.456 \times 10^{-7}\ \mathrm{GeV}^{-2} \simeq 2.5 \times 10^{-7}\ \mathrm{GeV}^{-2}. \tag{4.1}$$

Inversion of this bound on the coupling provides an estimate of the limiting energy scale for applicability of the model. Simple inversion ($g = 1$) suggests that $M \sim 2000\,\mathrm{GeV}$. It is a strongly held belief that the standard model itself will break down at an energy scale of or below this magnitude, so we are not concerned that the growth with beam energy will be unconstrained.

Finally, we note that, when we set the masses of the particles equal to zero for the $\mu + \mu \rightarrow \mu + \mu$ cross section, the result agrees with the ultra-high-energy calculation performed by Pilling [19]. This correspondence is a good indication of the consistency and correctness of our computations.

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