BÄCKLUND TRANSFORMATIONS AND NONLOCAL SYMMETRIES

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The theory of Lie groups gives a constructive method of constructing all point symmetries of differential equations. The classics of group analysis have emphasized the importance of developing methods of computing and using nonpoint symmetries ([1], p. 223; [2], §9). A partial solution of this problem is given in the theory of Lie-Bäcklund groups [3] which makes it possible to find all local (point and nonpoint) symmetries. In practice, however, equations are encountered which admit operators $X = f \partial / \partial u + \cdots$ whose coordinates $f$ depend not only on a finite number of local variables $x, u, u_1, \ldots$ (as in the case of Lie-Bäcklund groups) but also on expressions of the type of integrals of $u$. Such symmetries are called nonlocal. The main obstacle to directly carrying over the theory of Lie-Bäcklund groups to the case of nonlocal symmetries is the following circumstance. In the local theory essential use is made of the invariance of a space $\mathcal{A}$ of differential functions with respect to differentiation $D_x$. If together with $x, u, u_1, \ldots$ we now introduce “natural” nonlocal variables $u_{-1}, u_{-2}, \ldots$ (as is done, for example, in [4]), where $D_x(u_{-k}) = u_{-k+1}$, and a space $\mathcal{A}$ of functions of any finite number of local and nonlocal variables, then it turns out that the integration $D_x^{-1}(u^2) = uu_{-1} - u_1 u_{-2} + u_2 u_{-3} - \cdots \notin \mathcal{A}$. Therefore, special methods must be developed in order to constructively compute nonlocal symmetries.

In this note we propose a method of constructing nonlocal symmetries of differential equations based on Bäcklund transformations and the theory of Lie-Bäcklund groups. We use the notation of [5] (see also [3], §19) modified to the case of several differential variables: $x$ is the independent variable; $D_x$ is differentiation; $u = (u^1, \ldots, u^m)$ is a vector-valued differential variable with successive derivatives $u_1, u_2, \ldots$, i.e., variables such that $D_x(u_i) = u_{i+1}$, $u_0 = u$; $\mathcal{A}^m[x, u]$ is the space of differentiable vector-valued functions $f = (f_1, \ldots, f^m)$ with components $f^\alpha$ which are analytic functions of any finite number of variables $x, u, u_1, \ldots$; for any $f \in \mathcal{A}^m[x, u]$ we define the differential operators

$$ (f_*)^\alpha = \sum_{i \geq 0} \frac{\partial f}{\partial u_i^\alpha} D_x^i, \quad \alpha = 1, \ldots, m; $$

and for $f$ and $F$ in $\mathcal{A}^m[x, u]$ we define the Lie bracket $\{F, f\}$ as the vector in $\mathcal{A}^m[x, u]$ with components

$$ \{F, f\}^\alpha = F^\alpha \cdot f - f^\alpha \cdot F, \quad F^\alpha \cdot f = \sum_{\beta=1}^m (F^\alpha)^\beta f^\beta. $$

1. Quasilocal symmetries. We consider systems of evolution equations

(1) $u_t = F, \quad F \in \mathcal{A}^m[x, u];$

(2) $v_t = G, \quad G \in \mathcal{A}^m[y, v].$

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connected by a Bäcklund transformation of the type of a differential substitution

(3) \quad y = \varphi, \quad v = \Phi, \quad \text{where } \varphi \in \mathcal{A}^1[x, u], \quad \Phi \in \mathcal{A}^m[x, u].

Here we shall consider only differential substitutions of first order, i.e., the cases where \( \varphi \) and \( \Phi \) depend on the variables \( x, u, \) and \( u_1. \)

Suppose system (1) admits an algebra of local symmetries, i.e., a Lie-Bäcklund algebra

\( \mathcal{A}_F^m[x, u] = \{ f_u \in \mathcal{A}^m[x, u] : \partial f_u/dt - \{ F, f_u \} = 0 \}. \)

If after the action of the differential substitution (3) \( f_u \in \mathcal{A}_F^m[x, u] \) goes over into \( f_v \in \mathcal{A}_G^m[y, v] \), then \( f_u \) and \( f_v \) are connected by the transition formulas [5] (see also [3], §19)

(4) \quad D_x(\varphi)f^\alpha_v = (D_x(\varphi)f^\alpha_u - D_x(\Phi^\alpha_u)\varphi_u) \cdot f_u, \quad \alpha = 1, \ldots, m.

Extension of (4) to arbitrary \( f_v \) (in the general case \( f_v \notin \mathcal{A}_F^m[y, v] \)) leads to a special class of nonlocal symmetries which we call quasilocal. Computation of these nonlocal symmetries is realized on the basis of the following definition.

**DEFINITION.** A quasilocal symmetry of equations (2) associated with \( f_u \in \mathcal{A}_F^m[x, u] \) is a vector-valued function \( f_v \) which is connected with \( f_u \) by the transition formulas (4) and together with \( t, y, v, v_1, \ldots \) depends also on certain new variables \( Q^1, \ldots, Q^k \) so that

\[ \frac{\partial f_v}{\partial t} - \{ G, f_v \} + \sum_{s=1}^k \frac{\partial f_v}{\partial Q^s} Q^s_t = 0. \]

In computing the bracket \( \{ G, f_v \} \) we use differentiation \( D_y \) extended to the nonlocal variables \( Q^s \) in the same way as to the usual differential variables.

By inverting the sequence of arguments in this definition, it is possible to obtain quasilocal symmetries \( f_u \) for equation (1) associated with local symmetries \( f_v \) for (2). We note that the transition formulas (4) are also applicable in the case where both symmetries \( f_u \) and \( f_v \) are quasilocal.

The introduction of quasilocal symmetries makes it possible to approach the problem of group classification of differential equations possessing Bäcklund transformations in a new way. Here it is also necessary to enlarge the group of equivalences by the introduction of quasilocal equivalence transformations.

The essence of our approach is illustrated below with some examples from mechanics.

2. Equations of nonlinear heat conduction. We consider the sequence of equations

(\text{H}) \quad w_t = H(w_{xx}) \quad v = w_t, \quad v_t = h(v_x)v_{xx} \quad u = v_t, \quad u_t = (h(u)u_x)_x, \quad h = H',

which are connected with one another by the simplest Bäcklund transformations (by differentiation and integration) and describe, respectively, the inertia-free oscillations of a string in a fluid with nonlinear resistance, the process of filtration of a non-Newtonian fluid in a porous medium, and the propagation of heat in a nonlinear medium. The approach proposed in §1 makes it possible to construct all quasilocal symmetries for the sequence (\text{H}). For example, for the case \( H(w_{xx}) = \arctan(w_{xx}) \), proceeding from the operator \( \lambda_{\alpha} = v \partial/\partial x - x \partial/\partial v \) of the point group admitted by the middle equation of (\text{H}), we obtain the following series of associated symmetries: \( X_w \rightarrow X_v \rightarrow X_u \), where \( X_u = v \partial/\partial x - (1 + u^2) \partial/\partial n \) is an operator of nonlocal symmetry and \( X_w = w_x \partial/\partial x - (t + (x^2 - w^2)/2) \partial/\partial w \) is an operator of tangential symmetry.

The point group of equivalences for the equation of filtration is presented in [6]. Using Bäcklund transformations, it can easily be carried over to the groups of equivalences for the remaining terms of the sequence (\text{H}). Nonlocal and tangential transformations here arise in a natural way. In particular, the equivalence group of the filtration equation contains the change of variables \( x' = v, \quad v' = x \), which goes over into the nonlocal
transformation\(^{(1)}\) for the heat equation and into the tangential transformation \(x' = w_x, w' = xw_x - w\) for the first equation of (H).

3. The equations of gas dynamics. We consider the sequence

\[
\begin{align*}
(L) & \quad y = R_z, q = R_z^{-1} \\
(1) & \quad \rho = R_z \\
(E) & \quad \rho_t = -v \rho z - \rho v_x,
\end{align*}
\]

of equations of one-dimensional gas dynamics written in Lagrangian (L) and Euler (E) coordinate systems. The intermediate system (I) makes it possible to connect (L) with (E) by means of a sequence of differential substitutions of the form (3). Here \(B = B(p, q)\). For a polytropic gas \(B = \gamma p/q\) quasilocal symmetries occur for \(\gamma = -1\) (a Chaplygin gas) and for \(\gamma = 3\). The operators of the corresponding symmetries are presented in Table 1 where \(x\) (for (L)) and \(P, Q, R\) (for (E)) are nonlocal variables defined by the equations

\[
\begin{align*}
& x_y = q, \quad x_t = v; \quad P_z = 1/p, \quad P_t = -v/p; \\
& Q_z = R/p, \quad Q_t = -vR/p - t; \quad R_z = p, \quad R_t = -pv.
\end{align*}
\]

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>(L)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>(X_L = t \frac{\partial}{\partial t} - y \frac{\partial}{\partial p} - yq \frac{\partial}{\partial q})</td>
<td>(X_E = -Q \frac{\partial}{\partial x} + t \frac{\partial}{\partial v} - R \frac{\partial}{\partial p} + \frac{\rho R}{p} \frac{\partial}{\partial \rho})</td>
</tr>
<tr>
<td>&amp; (Y_L = \frac{\partial}{\partial p} + \frac{q}{p} \frac{\partial}{\partial q})</td>
<td>(Y_E = P \frac{\partial}{\partial x} + \frac{\partial}{\partial p} - \frac{\rho}{p} \frac{\partial}{\partial \rho})</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(Z_L = t^2 \frac{\partial}{\partial t} + (x - tv) \frac{\partial}{\partial v})</td>
<td>(Z_E = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + (x - tv) \frac{\partial}{\partial v})</td>
</tr>
<tr>
<td>&amp; (-3tp \frac{\partial}{\partial p} + tq \frac{\partial}{\partial q})</td>
<td>(-3tp \frac{\partial}{\partial p} - tp \frac{\partial}{\partial \rho})</td>
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BIBLIOGRAPHY


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\(^{(1)}\)The papers [7] and [8] are devoted to a direct proof of the fact that it is an equivalence transformation of the nonlinear heat equation.