Pattern formation in acoustic cavitation

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(Received 21 March 1994; accepted for publication 21 July 1994)

A new approach for the theoretical description of structure formation in acoustic cavitation is developed. The model consists of two coupled partial differential equations describing the spatiotemporal evolution of the sound field amplitude and the bubble concentration. Linear stability analysis and numerical simulations of the pattern formation are presented. The relation between this approach and streamer formation is discussed.

PACS numbers: 43.25.Yw, 43.25.Ts

INTRODUCTION

Sound waves of high intensity may rupture a liquid, giving rise to the phenomenon of acoustic cavitation. The cavities or cavitation bubbles thereby produced group themselves in a remarkable way into a branched structure on a scale much smaller than the wavelength of the incident sound field. The name "streamers" has been attached to the pattern, and also "acoustic Lichtenberg figures" because of the striking similarity with the electrical discharge pattern obtained centuries ago by Lichtenberg. Figure 1 gives an example of a bubble pattern as observed inside a cylindrical piezoelectric transducer operated under water at about 14 kHz. No theory exists yet to describe the formation of the filamentary structure. This paper is a first approach to this question starting from acoustic waves in a homogeneous distribution of microbubbles and investigating the stability of the configuration.

Due to (primary) Bjerknes forces, the bubbles move to specific locations of the acoustic wave. This motion, however, changes the spatial distribution of microbubbles, which by itself has a strong influence on the sound field due to the dependence of the speed of sound on the bubble concentration. This topic is addressed in Sec. I. In Sec. II we derive a partial differential equation for the amplitude of the sound field. It turns out that the dynamics of the amplitude can be described by a nonlinear Schrödinger equation where the potential is replaced by the concentration of bubbles. The differential equation for the bubble concentration is derived in Sec. III. In Secs. IV and V boundary conditions and constants of motion are discussed. Section VI contains a linear stability analysis for perturbations of plane waves and uniform solutions that provides a criterion for long wavelength instabilities. They may be interpreted as the reason for the pattern formation as observed in the numerical simulations that will be presented in Sec. VII. The numerical methods used for solving our system of two coupled partial differential equations are briefly summarized in the Appendix.

I. DEPENDENCE OF SOUND SPEED ON THE BUBBLE CONCENTRATION

Let \( \rho_l, \rho_g \) and \( \alpha_l, \alpha_g \) be the density and the volume concentration of the liquid and the gas, respectively. Then the density \( \rho \) of the two-phase mixture is given by

\[
\rho = \rho_l \alpha_l + \rho_g \alpha_g,
\]

with

\[
\alpha_l + \alpha_g = 1.
\]

After differentiation with respect to the pressure \( p \), Eq. (1) yields

\[
c^{-2} = c_l^{-2} \alpha_l + c_g^{-2} \alpha_g \frac{d \alpha_g}{dp},
\]

where \( c_l, c_g, \) and \( c \) denote the speed of sound in the liquid, in the (ideal) gas, and in the mixture, respectively, and are given by

\[
c_l^{-2} = \frac{d \rho_l}{dp} , \quad c_g^{-2} = \frac{d \rho_g}{dp} = \frac{\rho_g}{\kappa p_0} , \quad c^{-2} = \frac{dp}{d\rho},
\]

where \( \kappa \) is the polytropic exponent and \( p_0 \) is the static pressure. Let \( m_g \) be the mass of gas in a single bubble with volume \( V_g \) and \( N \) be the concentration of bubbles per unit volume. Here and in the following we assume for simplicity that all bubbles have the same size. Then

\[
m_g = \rho_g V_g , \quad \alpha_g = NV_g ,
\]

and, using conservation of mass,

\[
\frac{d \alpha_g}{dp} = \frac{d}{dp} (NV_g) = N m_g \frac{d}{dp} \left( \frac{1}{\rho_g} \right) = - \frac{\alpha_g}{\rho_g} c^{-2}
\]

is obtained. Equation (3) can be rewritten in the following form:

\[
c^{-2} = c_l^{-2} (1 + \epsilon n),
\]

with
\[ \varepsilon = \left( \frac{c^2 \rho_l}{\kappa \rho_0} - 1 \right) V_g N_0 \]  \tag{8} \\
and \\
\[ n = N/N_0. \]  \tag{9} 

The parameter \( \varepsilon \) equals approximately \( 10^{-2} \), if we assume the following values for the physical parameters: \( c_l \sim 10^3 \) m/s, \( \rho_l \sim 10^3 \) kg/m\(^3\), \( \kappa \sim 1 \), \( \rho_0 \sim 10^3 \) N/m\(^2\), \( V_g = 4/3 \pi R_0^3 \), \( R_0 \sim 10^{-5} \) m, and \( N_0 \sim 10^9 \) m\(^{-3}\) = \( 10^3 \) cm\(^{-3}\).

II. EVOLUTION OF THE SOUND FIELD AMPLITUDE

For describing the evolution of the pressure amplitude \( P \) of the sound field we start with the two-dimensional wave equation

\[ c^{-2} \frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}, \]  \tag{10} 

where \( c \), the speed of sound in the mixture, is given by Eq. (7). (We write \( c \) [see Eq. (7)] instead of \( c_l \) for the surrounding liquid because later we always consider the sound field as propagating in a two-phase medium.)

Let us consider the stability of plane acoustic waves that propagate along the \( x \) axis. In the case \( \varepsilon = 0 \) the exact solution of Eq. (10) for the wave propagation process may be written as

\[ P = \frac{1}{2} \left[ W_0 \exp\left[ i\omega \left( t - \frac{x}{c} \right) \right] + \text{c.c.} \right], \]  \tag{11} 

where \( W_0 \) is the constant complex wave amplitude (c.c. is the complex conjugate). For stability analysis we consider perturbations of \( W_0 \) distributed in the front of the plane wave (i.e., along the \( y \) axis). Therefore we approximate the solution of Eq. (10) in the following form:

\[ P = \frac{1}{2} \left[ W(T,Y) \exp\left[ i\omega \left( t - \frac{x}{c} \right) \right] + \text{c.c.} \right], \]  \tag{12} 

where \( T = \varepsilon t \) and \( Y = \varepsilon y \) are slow variables. Substituting the derivatives

\[ \frac{\partial^2 P}{\partial x^2} = -\left( \frac{\omega}{c} \right)^2 \frac{1}{2} \left[ W \exp\left[ i\omega \left( t - \frac{x}{c} \right) \right] + \text{c.c.} \right], \]  \tag{13} 

into Eq. (10) and neglecting the terms proportional to \( \varepsilon^2 \) yields with Eq. (7)

\[ \frac{2i\omega}{c^2} \frac{\partial W}{\partial T} = \frac{\partial^2 W}{\partial Y^2} + \left( \frac{\omega}{c} \right)^2 nW. \]  \tag{16} 

If we use the dimensionless variables \( \xi, \eta \), and \( w \),

\[ \xi = (1/2) \omega T, \quad \eta = kY, \quad w = W/W_0, \]  \tag{17} 

where

\[ k = \omega/c \]  \tag{18} 

is the wave number and \( w \) is the dimensionless amplitude of the sound field, we obtain the amplitude equation in the following form:

\[ \frac{\partial w}{\partial \xi} = \frac{2w}{\partial \eta^2} + \frac{\omega^2}{c^2} nW. \]  \tag{19} 

Equation (19) is a nonlinear Schrödinger equation with the potential being replaced by the distribution of the concentration of bubbles \( n \).

III. EVOLUTION OF THE BUBBLE CONCENTRATION IN THE SOUND FIELD

All bubbles with volume \( V_g \) experience a force

\[ \mathbf{F} = -V_g \nabla P, \]  \tag{20} 

where \( \nabla P \) is the pressure gradient. If \( P \) and \( V_g \) vary in time with high frequency, it is possible to calculate the time average of the force. This primary Bjerknes force can be written as

\[ \mathbf{F}_b = -\langle V_g(t) \nabla P(x,y,t) \rangle, \]  \tag{21} 

where the angle brackets \( \langle \cdot \rangle \) denote the time average (over one period \( 2\pi/\omega \) of the oscillation of \( P \)). In our case, the pressure field may be described by the following equation:

\[ P = P_0 + \frac{1}{2} \left[ (U + iV) \exp\left[ i\omega \left( t - \frac{x}{c} \right) \right] \right. \]

\[ + (U - iV) \exp\left[ -i\omega \left( t - \frac{x}{c} \right) \right] \]

\[ = P_0 + U \cos\omega \left( t - \frac{x}{c} \right) - V \sin\omega \left( t - \frac{x}{c} \right), \]  \tag{22} 

where

\[ W(\xi, \eta) = U(\xi, \eta) + iV(\xi, \eta). \]  \tag{23} 

Now we shall consider a bubble located at a point \((x,y)\) in the acoustic field and oscillating far below its resonance frequency \( \omega_r \). For small deviations \( r = R - R_0 \) of the radius...
of the bubble \( R \) from its value at equilibrium \( R_0 \) we obtain the following equation of motion:

\[
\ddot{r} + \frac{\omega^2 r}{\dot{r}} = \frac{1}{\rho R_0} \left[ U(\xi, \eta) \cos \left( \omega \left( t - \frac{x}{c} \right) \right) \right. \\
- V(\xi, \eta) \sin \left( \omega \left( t - \frac{x}{c} \right) \right) \left. \right],
\]

(24)

Here we take \( \rho \) [see Eq. (1)] instead of \( \rho_0 \) as above \( c \) instead of \( c_1 \) because we consider the bubble as being located in a two-phase medium.

The solution of this equation is

\[
r = \frac{1}{\rho R_0 (\omega^2 - \omega^2)} \left[ U(\xi, \eta) \cos \left( \omega \left( t - \frac{x}{c} \right) \right) \right. \\
- V(\xi, \eta) \sin \left( \omega \left( t - \frac{x}{c} \right) \right) \left. \right],
\]

(25)

because \( U \) and \( V \) are functions that change slowly in time and space. Then the oscillations of the bubble volume are given by

\[
V_B(t) = \frac{4}{3} \pi R^3(t) \\
= \frac{4}{3} \pi R_0^3 \left( 1 + \frac{r(t)}{R_0} \right)^3 \\
= V_0 \left( 1 + \frac{3}{R_0} r(t) \right)
\]

(26)

The pressure gradient \( \nabla P = (\partial P/\partial x, \partial P/\partial y) \) is given by

\[
\frac{\partial P}{\partial x} = \frac{\omega}{c} \left[ U \sin \left( \omega \left( t - \frac{x}{c} \right) \right) + V \cos \left( \omega \left( t - \frac{x}{c} \right) \right) \right],
\]

(27)

\[
\frac{\partial P}{\partial y} = k \sqrt{\varepsilon} \left[ \frac{\partial U}{\partial \eta} \cos \left( \omega \left( t - \frac{x}{c} \right) \right) - \frac{\partial V}{\partial \eta} \sin \left( \omega \left( t - \frac{x}{c} \right) \right) \right].
\]

If we insert Eqs. (26) and (27) into Eq. (21) and average over one period \( 2\pi/\omega \), we obtain the primary Bjerknes force \( F_B \):

\[
F_B = - \left( 0, \gamma_1 \frac{\partial (|W|^2)}{\partial \eta} \right), \quad \gamma_1 = \frac{3V_0}{4\rho R_0^3 (\omega^2 - \omega^2)} k \sqrt{\varepsilon}.
\]

(28)

The equation for the slow drift of bubbles in the liquid is

\[
F_B + F_t = 0,
\]

(29)

where \( F_t \) equals the friction force. Now we assume that \( F_t \) is directly proportional to the velocity of the drift of the bubble \( \dot{F}_t = -k_v \dot{V} \) and thus obtain

\[
\dot{V} = \left( 1/k_v \right) F_B.
\]

(30)

Using Eqs. (28) and (30), it is possible to rewrite the conservation law for the bubble concentration

\[
\frac{\partial N}{\partial t} + \text{div}(NV) = 0,
\]

(31)

in the following form:

\[
\frac{\partial n}{\partial \xi} = \gamma \frac{\partial}{\partial \eta} \left[ n \frac{\partial}{\partial \eta} (|W|^2) \right],
\]

(32)

with

\[
\gamma = \frac{3V_0 k^2 |W_0|^2}{2\rho R_0^3 (\omega^2 - \omega^2)} k \sqrt{\varepsilon}.
\]

(33)

It is well known that microbubbles cannot exist for a long time without an acoustic field (see Ref. 9). Applying theoretical results,\(^{10}\) one can show that air bubbles the size of a few micrometers in water will dissolve in a few seconds. Let us assume that the volume of the bubbles decreases exponentially in time. Since we consider only bubbles of fixed size, this effect is taken into account by an exponentially decreasing number of bubbles that is given by the differential equation

\[
\frac{\partial N}{\partial t} = - \frac{N}{\tau_1},
\]

(34)

where \( \tau_1 \) is the characteristic time of dissolution of microbubbles.

When an acoustic field of sufficiently high amplitude is switched on, it can stop the dissolution and supports the formation of microbubbles. It is easy to estimate\(^{11}\) that the energy flow into the bubble for one period of oscillations is directly proportional to the square of the pressure amplitude of the external field. This fact can be phenomenologically taken into account by including an additional term

\[
f(|W|^2) = \delta |W|^2 + \cdots
\]

(35)

in Eq. (34) to describe the generation of bubbles by the acoustic field. Then Eq. (34) can be written as

\[
\frac{\partial N}{\partial t} = - \frac{N - \delta |W|^2}{\tau_1},
\]

(36)

with \( \delta \) being the bubble production parameter. It should be emphasized that for the derivation of Eq. (36) the effects connected with threshold phenomena (e.g., quasistatic Blake threshold pressure)\(^{12}\) have been omitted, because the main aim of the present paper is to investigate the self-organizing behavior of bubble fields above the cavitation threshold.

Adding the concentration growth and decay term of Eq. (36) to Eq. (32) yields the desired partial differential equation for the evolution of the bubble concentration in a sound field. The dimensionless form of this equation, when we choose as normalizing constant \( N_0 = \delta |W_0|^2 \), is

\[
\frac{\partial n}{\partial \xi} = \gamma \frac{\partial}{\partial \eta} \left[ n \frac{\partial}{\partial \eta} (|W|^2) \right] - \frac{n - |W|^2}{\tau},
\]

(37)

where

\[
\tau = \left( \frac{\omega}{2} \right) \tau_1.
\]

(38)
The parameter \( \tau \) is difficult to estimate exactly. In any case, however, it has to be much larger than the period of the acoustic oscillation. Using Stokes’s law \( k_n = 6 \pi \mu R_0 \) and Eq. (28), we obtain for \( \gamma \) the expression

\[
\gamma = \frac{\omega |W_0|^2}{3 \rho \mu (\omega^2 - \omega^2) c^2}.
\]  

(39)

With the numerical values \( \omega = 100 \text{ kHz} \), \( \omega_0 = 1000 \text{ kHz} \), \( \mu = 10^{-3} \text{ kg/m s} \), \( \rho = 1000 \text{ kg/m}^3 \), \( c = 1000 \text{ m/s} \), and \( |W_0|^2 = 3 \times 10^{10} \text{ kg}^{3/2} \text{ m}^2 \), \( \gamma \) can be estimated to be \( \gamma = 0.001 \).

The value of \( \gamma \) increases very fast when the frequency \( \omega \) approaches the resonant frequency \( \omega_0 \).

Finally we want to note that for \( \tau = 0 \) Eq. (37) implies \( n = |w|^2 \), and therefore Eq. (19) becomes the ordinary nonlinear Schrödinger equation

\[
\frac{\partial w}{\partial \xi} = \frac{\partial^2 w}{\partial \eta^2} + |w|^2 w.
\]  

(40)

This equation occurs in a variety of physical contexts, for instance, in light propagation in optical fibers.\(^{13}\) It has also been derived for modulation waves in bubble–liquid mixtures with a fixed number of bubbles by Gumerov,\(^{14}\) who used the method of multiple scales.

### IV. BOUNDARY CONDITIONS

The boundary condition for the amplitude \( w \),

\[
\frac{\partial w}{\partial \eta} (0, \xi) = \frac{\partial w}{\partial \eta} (1, \xi) = 0,
\]  

(41)

describes the reflection from the lateral walls of the channel. Using this condition in Eq. (37) leads to an ordinary differential equation for the evolution of the concentration \( n \) of bubbles at the boundaries \( \eta = 0 \) and \( \eta = 1 \):

\[
\frac{\partial n}{\partial \xi} = -\left( \frac{1}{\tau} - \gamma \frac{\partial^2}{\partial \eta^2} (|w|^2) \right) n + |w|^2 \frac{\partial w}{\partial \xi}.
\]  

(42)

To rewrite this differential equation we derive from the boundary condition [Eq. (41)] for \( w \) the equation

\[
\frac{\partial}{\partial \eta^2} (|w|^2) = w \frac{\partial^2 w}{\partial \eta^2} + w^* \frac{\partial^2 w}{\partial \eta^2}.
\]  

(43)

Multiplying Eq. (19) by \( w^* \) and summing the result with its complex conjugate, we get

\[
w \frac{\partial^2 w^*}{\partial \eta^2} + w^* \frac{\partial^2 w}{\partial \eta^2} = -2 n |w|^2 i \left( w^* \frac{\partial w}{\partial \xi} - w \frac{\partial w^*}{\partial \xi} \right).
\]  

(44)

Combining Eq. (43) and Eq. (44), we can rewrite the equation for the boundary values of \( n \) in the following form:

\[
\frac{\partial n}{\partial \xi} = -\left( \frac{1}{\tau} + 2 \gamma \text{Im} \left( \frac{w^* \frac{\partial w}{\partial \eta} + 2 \gamma |w|^2 n}{\partial \eta} \right) n + \frac{|w|^2}{\tau} \right. \tag{\eta = 0, 1). \tag{45}
\]

The solution of this equation is the time-dependent boundary condition for Eq. (37).

### V. CONSTANTS OF MOTION

It is easy to show that the energy of the sound field amplitude,

\[
E = \int_0^1 |w(\eta, \xi)|^2 \, d\eta,
\]  

(46)

is a constant of motion that depends only on the initial distribution of the amplitude \( w = w(\eta, 0) \). The temporal evolution of the total number of bubbles,

\[
M = \int_0^1 n(\eta, \xi) \, d\eta,
\]  

(47)

can be described by the following ordinary differential equation:

\[
\dot{M} = (E - M) / \tau.
\]  

(48)

Equation (48) is obtained by integrating Eq. (37) in space, taking into account the boundary conditions [Eq. (41)] for the amplitude. When the initial distribution of bubbles \( n(\eta, 0) \) is chosen to be equal to the initial energy density of the acoustic field \( |w(\eta, 0)|^2 \), then \( M(0) = E(0) = E_0 \), and the total number of bubbles \( M \) has to be constant in time, \( M(\xi) = E_0 \). This fact has been used for controlling the accuracy of the numerical method. For all results presented in this paper the deviations of \( E \) and \( M \) from their initial values were smaller than \( 1\% \). If \( M(0) \) is different from \( E_0 \), then \( M(\xi) \) converges to this stable equilibrium value, as can be seen directly from Eq. (48).

### VI. STABILITY ANALYSIS

In this section the stability of uniform solutions of Eqs. (19) and (37) with respect to small perturbations is analyzed. The set of equations considered is repeated for convenience:

\[
\frac{i}{\tau} \frac{\partial w}{\partial \xi} = \frac{\partial^2 w}{\partial \eta^2} + n w,
\]  

(49)

\[
\frac{\partial n}{\partial \xi} = \gamma \frac{\partial}{\partial \eta} \left( n \frac{\partial}{\partial \eta} (|w|^2) \right) - n |w|^2 \frac{\partial w}{\partial \xi}.
\]  

(50)

If we write the complex amplitude \( w \) in the form

\[
w = A \exp(i \Theta),
\]  

(51)

where \( A = A(\xi, \eta) \) and \( \Theta = \Theta(\xi, \eta) \) are real functions, we obtain

\[
-A \frac{\partial A}{\partial \xi} = \frac{\partial^2 A}{\partial \eta^2} - A \left( \frac{\partial A}{\partial \eta} \right)^2 + A n,
\]  

(52)

\[
\frac{\partial A}{\partial \xi} = 2 \frac{\partial A}{\partial \eta} \frac{\partial \Theta}{\partial \eta} + A \frac{\partial^2 \Theta}{\partial \eta^2},
\]  

(53)

\[
\frac{\partial n}{\partial \xi} = \gamma \frac{\partial}{\partial \eta} \left( n \frac{\partial}{\partial \eta} (A^2) \right) - n A^2 \frac{\partial A}{\partial \xi}.
\]  

(54)

It is easy to verify that \( A = A_0 = \text{const.} , \quad n = A_0^2, \quad \Theta = -A_0^2 \xi \)

(55)

is a uniform \((\partial / \partial \eta = 0) \) solution of this system. The evolution of a small perturbation of this uniform solution,
\[ A = \dot{A}_0 + \ddot{A}, \quad \Theta = -A_0^2 + \dot{\Theta}, \quad n = A_0^2 + \dot{n}, \tag{56} \]

is given by the following linearized equations:

\[ 0 = \dot{A}_0 + \frac{\partial^2 \dot{A}}{\partial \xi^2} + A_0 \dot{n}, \tag{57} \]

\[ 0 = \frac{\partial \dot{\Theta}}{\partial \xi} - A_0 \frac{\partial^2 \dot{\Theta}}{\partial \eta^2}, \tag{58} \]

\[ 0 = \frac{\partial \dot{n}}{\partial \xi} - 2 \gamma A_0^3 \frac{\partial^2 \dot{A}}{\partial \eta^2} + \frac{1}{\tau} (\dot{n} - 2 A_0 \ddot{A}). \tag{59} \]

Now let us consider the evolution of a periodic perturbation with wavelength \( L = 2 \pi / K \) that can be written as

\[ \begin{pmatrix} \dot{A} \\ \dot{\Theta} \\ \dot{n} \end{pmatrix} = \begin{pmatrix} A_1 \\ \Theta_1 \\ n_1 \end{pmatrix} \exp(\sigma \xi + iK \eta). \tag{60} \]

The stability of the uniform solution depends on the sign of the real part of the growth rate coefficient, \( \sigma \). To compute \( \sigma \) we substitute the perturbation Eq. (60) into the linearized Eqs. (57)–(59) and thus obtain the following cubic polynomial in \( \sigma \):

\[ f(\sigma) = K^4 - 2A_0^2 K^2 (1 - \tau \gamma A_0^2 K^2) + \tau K^4 \sigma + \sigma^2 + \tau \sigma^3. \tag{61} \]

According to the Hurwitz criterion all roots \( \sigma \) of this polynomial possess negative real parts if and only if the following quantities are positive:

\[ D_1 = K^4 - 2A_0^2 K^2 (1 - \tau \gamma A_0^2 K^2), \tag{62} \]

\[ D_2 = 2 \tau A_0^2 K^2 (1 - \tau \gamma A_0^2 K^2), \tag{63} \]

\[ D_3 = \tau D_2. \tag{64} \]

Thus, the perturbation is stable for \( K^2 \) values in the range

\[ \frac{2A_0^2}{1 + 2 \tau \gamma A_0^2} < K^2 < \frac{1}{\tau \gamma A_0^2}, \tag{65} \]

i.e., there exists a long-wavelength instability given by the threshold \( K_1 \) and a short-wavelength instability with threshold \( K_2 \). In Fig. 2 the threshold \( K_1 \) is plotted versus the square of the amplitude \( A_0^2 \). It is easy to see that there exists a critical amplitude

\[ \dot{A}_0^2 = 1/\sqrt{2 \tau \gamma} \tag{66} \]

for which \( K_1 \) attains its maximum value \( \dot{K}_1 \). Thus we obtain a maximal range of wave numbers corresponding to unstable perturbations with long wavelength:

\[ K^2 < K_1^2 = 1/\sqrt{2 \tau \gamma}. \tag{67} \]

This long-wavelength instability may be interpreted as the reason for the occurrence of nonlinear structures, which will be discussed in Sec. VII.

![FIG. 2. Criterion for long-wavelength instability for \( \tau=1 \) and \( \gamma=0.02 \). The dotted line gives the stability threshold for the nonlinear Schrödinger equation [Eq. (40)].](image)

VII. RESULTS OF NUMERICAL SIMULATIONS

In this section we present results of numerical simulations using Eqs. (37) and (19) for the evolution of the bubble concentration and the amplitude of the sound field, respectively.

We have simulated the evolution of small perturbations of acoustic waves propagating along a channel with reflecting boundaries. The initial condition consists of a uniform distribution of the amplitude of the acoustic field and the concentration of bubbles that is perturbed by a cosine function with small amplitude. For simplicity we investigate only perturbations with wavelength equal to the distance between the lateral walls of the channel that is given by

\[ \omega(\eta, 0) = A_0 + A_1 \cos(\eta), \quad \eta(\eta, 0) = |\omega(\eta, 0)|^2, \tag{68} \]

with \( A_0 = 1 \) and \( A_1 = -0.1 \) (i.e., the case \( K = 1, \eta \in [0, 2\pi] \)).

For solving the partial differential equations a finite-difference scheme was used that is described in more detail in the Appendix. The spatial grid consisted of 257 points, and the size of the time steps was \( \Delta t = 0.1(\Delta y)^2 \), with \( \Delta y = 2\pi / 257 \). In the following figures the spatial interval has been normalized by \( 2\pi \) to \([0, 1] \).

Our model contains two significant parameters, \( \tau \) and \( \gamma \). The parameter \( \tau \) is the dimensionless time of dissolution of microbubbles, and \( \gamma \) is the dimensionless characteristic parameter of the Bjerknes force [Eq. (39)].

For systematically investigating the influence of \( \tau \) and \( \gamma \) on the evolution of structures we start from the limit case \( \tau=0 \), where the process of dissolution and generation takes place without time delay. In this case the solution does not depend on \( \gamma \), and the model is reduced to the ordinary nonlinear Schrödinger equation (40). Figure 3 shows that small initial perturbations of amplitude grow and start to oscillate. This means that the limit case \( \tau=0 \) cannot be used for an explanation of structure formation processes in cavitation dy-
namics. After a few oscillations the location of the maximum becomes unstable and moves to one of the boundaries, where it is reflected several times. This process is shown in Fig. 3(a) with a single reflection and in Fig. 3(b), in the form of a contour plot, for a longer time, demonstrating repeated reflections. This instability is the result of symmetry breaking, and the same series of reflections may be observed at the other boundary.

Qualitatively similar behavior occurs for relatively small values of $\tau$. After a few oscillations the maximum moves to the right or left wall and stays at the boundary finally without reflections. Increasing $\tau$ further leads to a stabilization of the structure. In Fig. 4 the result for $\tau=0.1$ is shown. Figure 4(a) gives the bubble concentration and Fig. 4(b) the sound field

FIG. 3. (a) Evolution of the amplitude of the sound field $|w|$ for $\tau=0$ and $\gamma=0$. (b) Contour plot of the evolution of the amplitude of the sound field $|w|$ for $\tau=0$ and $\gamma=0$.

FIG. 4. (a) Evolution of the bubble concentration $n$ for $\tau=0.1$ and $\gamma=0$. (b) Evolution of the amplitude of the sound field $|w|$ for $\tau=0.1$ and $\gamma=0$. (c) Comparison of the asymptotic solution $|w|$ for $\tau=0.1$ and $\gamma=0$ [see (b)] with the shape of a soliton (dotted line).
amplitude. The oscillations of the pulse are now damped and the pulse shape converges to a solitonlike structure located in the middle of the channel. Figure 4(c) shows a comparison of the final shape of the solution for $|w|$ for $\tau=0.1$ with the exact soliton solution $a \cdot \text{sech}(a\tau/\sqrt{2})$ of the nonlinear Schrödinger equation, where $a$ is the amplitude of the structure. When $\tau$ is increased furthermore, the transient oscillations become more strongly damped.

Now we introduce the influence of the Bjerknes forces by increasing the parameter $\gamma$. For small $\gamma$ the transient and the shape of the final structure are qualitatively the same. However, the width of the solitonlike structure increases, and its amplitude decreases. For larger values of $\gamma$ the final structure becomes unstable, because the Bjerknes forces divide it into several new structures. This process is shown in Figs. 5 and 6 for $\gamma=0.015$. During the first stage of the evolution the amplitude $|w|$ and the concentration $n$ are in phase (Figs. 5 and 6). Then, however, both distributions, $n$ and $|w|$, undergo several transitions where three, five, seven, nine, and finally eleven peaks occur. This can best be seen in the contour plot of the sound field amplitude $|w|$ shown in Fig. 6(b). Further computations until $t=3000$ indicate that the structure consisting of eleven bubble clusters is stable. In contrast to the first stage of the evolution, for all succeeding stages the locations of the emerging bubble clusters (maxima of $n$) coincide with minima of the sound field amplitude. A comparison of Figs. 5 and 6 shows that all transitions first become visible in the sound field amplitude $|w|$ and that the bubble concentration $n$ changes later. When the sharp peaks occur, in physical reality effects become dominant that are not included in the model (e.g., coalescence of bubbles and secondary Bjerknes forces). Simulations using spatial grids with higher resolution show that the peaks always contract until they cover only a few grid points. The reason for this behavior is the fact that the model in its present form does not contain any mechanism that prevents the peaks of the density $n$ from growing indefinitely. In our examples, however, maxima with constant height occur. This is due to the finite numerical resolution. Tests with different numerical parameters of time step size or spatial resolution show, however, that the first stage of the evolution is very robust and is reproduced in all cases. The transition to the peaks shown in Fig. 5 and the number and size of the maxima, however, turned out to be rather sensitive to numerical details. We suppose that during this transition the short-wave instability discussed in Sec. VI might be responsible for the strong oscillations that can be seen, for example, in Fig. 5. In any case in all our numerical simulations we obtained the emergence of local bubble clusters with high concentrations.

VIII. DISCUSSION

A one-dimensional model for pattern formation in acoustic cavitation is developed. It consists of an equation
for the evolution of the bubble concentration in an acoustic field [Eq. (37)] and a nonlinear Schrödinger equation for the amplitude of the acoustic field [Eq. (19)], where the potential is replaced by the distribution of bubbles. Linear stability analysis of uniform configurations of bubbles shows a short- and a long-wavelength instability. For small amplitudes of the acoustic wave the latter is quite similar to the long-wavelength instability for the nonlinear Schrödinger equation. The stability of large amplitudes, however, differs strongly from the nonlinear Schrödinger case due to primary Bjerknes forces (see Fig. 2). This difference leads to a new kind of nonlinear structure formation that was observed in numerical simulations and may be interpreted as a “self-concentration” of bubbles in the sound field, yielding strongly localized clusters.

The structures may be interpreted as the first step to the formation of streamers in the following way. The concentration inside of the structures formed is much higher than the initial distribution of bubbles. Therefore, after the formation of structures, secondary Bjerknes forces start to become important and have to be included. These secondary Bjerknes forces lead to coalescence of microbubbles. To describe this process it is necessary to generalize the model, taking into account secondary Bjerknes forces and a distribution of bubble sizes. Due to the coalescence of microbubbles and rectified diffusion, the structures become visible. These objects will be called streamers of the first generation in the following. In Fig. 7 such a streamer (denoted by $S_1$) is shown schematically. The energy of the external sound field is concentrated mainly in the streamer, and its amplitude outside is very small. Therefore, due to the high concentration of bubbles in the streamer, the amplitude of secondary acoustic waves radiated by these bubbles is comparable with the external sound field outside. In Fig. 7 wave vectors of the external and the radiated sound field are denoted by $k_e$ and $k_r$, respectively. The resulting wave vector $k$ gives the direction of the total acoustic field outside the streamer. In some sense we obtain a similar problem of stability for the new wave front $F$. This may lead to new structure formation in the direction of $k$, resulting in streamers of the second generation, $S_2$. Interaction of streamers of both generations can change the geometry of the structure. Furthermore, the repetition of this process might produce fractal structures.

Future development of this approach should take into account secondary Bjerknes forces, rectified diffusion, coalescence, and destruction of bubbles of different sizes and the nonlinear character of bubble oscillations. Furthermore, to compare the theoretical approach with the experimental results (see Fig. 1 and Ref. 7) it is necessary to consider the case of standing waves in a two-dimensional geometry.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft (Sonderforschungsbereich 185—Nichtlineare Dynamik) and the Stabsabteilung Internationale Beziehungen (Cooperation with the Russian Federation). Furthermore, we would like to thank C. Scheffczyk, M. Wiesenfeldt, and the other members of the Nonlinear Dynamics Group at the Institut für Angewandte Physik, Technische Hochschule Darmstadt, for many valuable discussions.

APPENDIX: NUMERICAL INTEGRATION METHOD

In this appendix we discuss the numerical scheme for solving the following system of partial differential equations:

$$
\frac{\partial w}{\partial t} = i \frac{\partial^2 w}{\partial y^2} + nw,
$$

(A1)

$$
\frac{\partial n}{\partial t} = \gamma \frac{\partial}{\partial y} \left( n \frac{\partial}{\partial y} \left| w \right|^2 \right) - \frac{n \left| w \right|^2}{\tau}
$$

(A2)

for $0 < y < 1$ and $t > 0$. (In this appendix we use the standard variables $y$ and $t$ for space and time.)

To solve the amplitude equation (A1) we modified a numerical algorithm for the linear Schrödinger equation for the nonlinear case:

$$
\frac{w_{j}^{n+1} - w_{j}^{n}}{\Delta t} = \frac{1}{2(\Delta y)^2} \left[ w_{j-1}^{n+1} - 2w_{j}^{n+1} + w_{j+1}^{n+1} + w_{j-1}^{n+1} - 2w_{j}^{n+2} + w_{j+1}^{n+2} \right] + (1/2)n_{j}^{n}(w_{j}^{n+1} + w_{j}^{n})
$$

(A3)

where $w_{j}^{n} = w(y_{j}, t_{n})$, $n_{j}^{n} = n(y_{j}, t_{n})$, $y_{j} = (j-1)\Delta y$, and $t_{n} = n\Delta t$, with $j = 1, \ldots, J$ and $n = 0, 1, 2, \ldots$. For numerical computations it is necessary to rewrite this system of linear equations in tridiagonal form:

$$
w_{j-1}^{n+1} + B_{j}^{n}w_{j}^{n+1} + w_{j+1}^{n+1} = R_{j}^{n} \quad (j = 2, \ldots, J - 1),
$$

(A4) with

$$
B_{j}^{n} = -2 - a + bn_{j}^{n},
$$

(A5)

$$
R_{j}^{n} = -w_{j+1}^{n} - w_{j-1}^{n} + (2 - a - bn_{j}^{n})w_{j}^{n},
$$

(A6)

where

$$
a = \frac{2(\Delta y)^2 i}{\Delta t}, \quad b = (\Delta y)^2.
$$

(A7)

The boundary condition Eq. (41) can be approximated by $w_{1}^{n+1} = w_{2}^{n+1}$ and $w_{J}^{n+1} = w_{J-1}^{n+1}$. To solve Eq. (A2) for the base...
density of bubbles we used the following discretization scheme:

\[
\frac{n^{n+1}_{j} - n^{n}_{j}}{\Delta t} = \frac{\gamma}{2(\Delta y)^2} \left\{ \frac{n^{n+1}_{j+1} + n^{n+1}_{j}}{2} \left( |w^{n+1}_{j+1}|^2 - |w^{n+1}_{j}|^2 \right) - \frac{n^{n+1}_{j+1} + n^{n+1}_{j-1}}{2} \left( |w^{n+1}_{j+1}|^2 - |w^{n+1}_{j-1}|^2 \right) \right. \\
- \frac{n^{n+1}_{j} + n^{n+1}_{j-1}}{2} \left( |w^{n+1}_{j}|^2 - |w^{n+1}_{j-1}|^2 \right) \\
+ \frac{n^{n}_{j+1} + n^{n}_{j}}{2} \left( |w^{n}_{j+1}|^2 - |w^{n}_{j}|^2 \right) \\
- \frac{n^{n}_{j} + n^{n}_{j-1}}{2} \left( |w^{n}_{j}|^2 - |w^{n}_{j-1}|^2 \right) \right\} \\
+ \frac{1}{2\tau} \left( |w^{n+1}_{j}|^2 + |w^{n+1}_{j}|^2 \right) - \frac{1}{2\tau} \left( n^{n+1}_{j+1} + n^{n}_{j} \right),
\]

(A8)

The tridiagonal form of this system is the following:

\[
A^{n+1}_{j} n^{n+1}_{j} + B^{n+1}_{j} n^{n+1}_{j} + C^{n+1}_{j} n^{n+1}_{j} = R^{n+1}_{j} \quad (j = 2, \ldots, J-1),
\]

(A9)

where

\[
A^{n}_{j} = c \left( |w^{n}_{j+1}|^2 - |w^{n}_{j}|^2 \right),
\]

(A10)

\[
B^{n}_{j} = c \left( |w^{n}_{j+1}|^2 - 2|w^{n}_{j+1}|^2 + |w^{n+1}_{j}|^2 \right) - 1 - d,
\]

(A11)

\[
C^{n}_{j} = c \left( |w^{n}_{j+1}|^2 - |w^{n}_{j}|^2 \right),
\]

(A12)

\[
R^{n}_{j} = -n^{n}_{j} - c \left[ (n^{n}_{j+1} + n^{n}_{j}) (|w^{n}_{j+1}|^2 - |w^{n}_{j}|^2) + (n^{n}_{j} + n^{n}_{j-1}) \right] \times (|w^{n}_{j-1}|^2 - |w^{n}_{j-1}|^2) - d \left( |w^{n}_{j+1}|^2 + |w^{n}_{j}|^2 \right) + n^{n}_{j},
\]

(A13)

and

\[
c = \frac{\gamma \Delta t}{4(\Delta y)^2}, \quad d = \frac{\Delta t}{2\tau}.
\]

(A14)

The equation for the boundary values of \( n \) [Eq. (45)] for the time interval \( t \in [t_n, t_{n+1}] \) was approximated by the following differential equation:

\[
\frac{dn}{dt} = -\left[ \frac{1}{\tau} + \gamma \text{Im} \left( \frac{w^{n+1}_{1} + w^{n}_{1}}{w^{n+1}_{1} - w^{n}_{1}} \right) \right]
\]

\[
+ \gamma \left( |w^{n+1}_{1}|^2 + |w^{n}_{1}|^2 \right) n + \frac{1}{2\tau} \left( |w^{n+1}_{1}|^2 + |w^{n}_{1}|^2 \right),
\]

(A15)

where \( w_{1,j} \) are the boundary values of the amplitude at the left (\( w_{1} \)) and at the right (\( w_{j} \)) sides. This equation was solved using a fourth-order Runge–Kutta scheme for the initial conditions \( n(t=0) = n^{n}_{j} \) (\( j = 1, J \)) and yields the boundary values for the next time step \( n(t_{n+1}) = n^{n+1}_{j} \) (\( j = 1, J \)).