Structure Formation in Cavitation Bubble Fields

U. PARLITZ,* C. SCHEFFCZYK,† I. AKHATOV‡ and W. LAUTERBORN*

*Institut für Angewandte Physik, TH Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany.
†Arbeitsgruppe ‘Nichtlineare Dynamik’ der MPG an der Universität Potsdam, An Neuen Palais, Gebäude 19,
D-14469 Potsdam, Germany. and ‡Department of Continuous Media Mechanics, Bashkir State University, 32
Frunze Str., Ufa 450074, Russia

Abstract — Two approaches for modelling the formation of filamentary structures in cavitation bubble
fields are presented. The first one describes the interaction of the sound field and the distribution of
microbubbles in terms of a set of two coupled partial differential equations that determine the
evolution of the sound-field amplitude and the bubble density. The second approach consists of a
quasideterministic aggregation model, where the bubbles are treated as pulsating particles which
experience radiation forces due to the sound-fields radiated from the other pulsating bubbles. Results
of numerical simulations are presented for both models. The validity and limitations of both
approaches are discussed.

1. INTRODUCTION

Acoustic cavitation occurs when a liquid is subjected to sound of high intensity so that it
may rupture and form cavities [1–3]. Under suitable conditions the cavities or cavitation
bubbles group themselves in a remarkable way to a branched structure (so called
streamers) that is similar to electrical discharge patterns [4]. An experimental arrangement
for the generation of streamers is shown in Fig. 1. Figure 2 gives an example of a bubble
pattern as observed inside a cylindrical piezoelectric transducer operated in water at about
13 kHz. In the following we present two approaches for modelling this structure formation
process. The first ansatz consists in a continuous model for the interaction of a plane
acoustic wave and an initially smooth distribution of microbubbles in a narrow channel. In
this case the evolution of the sound-field amplitude and of the bubble concentration can be
described by two coupled partial differential equations. Numerical simulations show that
instabilities of flat bubble distributions and the formation of microbubble clusters may
occur. In physical reality, at locations of high bubble concentration the microbubbles will
coalesce and form larger bubbles, as can be observed in the experiment. These visible
bubbles are generated at specific locations in the liquid and may be considered as the
starting point of the second modelling approach. There, the bubbles are treated as particles
that interact with the sound field radiated by the cavitation cylinder (primary Bjerknes
forces) and with the sound fields radiated from the other oscillating bubbles (secondary
Bjerknes forces). The secondary Bjerknes forces are proportional to $1/r^2$, where $r$ is the
distance between two bubbles. To investigate whether it is in principle possible to explain
the observable (fractal) structures by this $1/r^2$-force, a simple quasideterministic aggrega-
tion model is investigated. This model is easy to modify and allows simulations with a large
number of particles.
Fig. 1. The experimental arrangement used for streamer investigation consists of a cylindrical piezoelectric transducer operated under water at about 13 kHz.

Fig. 2. Bubble pattern as observed inside the cylinder looking along the axis.
2. CONTINUOUS MODEL FOR BUBBLE-FIELD DYNAMICS

2.1. A one-dimensional model: plane wave in a channel

As a first step towards the modelling of the full 3D-problem investigated experimentally we consider in the following:

- a plane acoustic wave propagating along a (narrow) channel (x-axis in Fig. 3)
- reflecting boundaries (the walls of the channel)
- a concentration \( n \) of bubbles of a fixed size that:
  oscillate harmonically due to the external sound field
  experience the (primary) Bjerknes force \( \mathbf{F}_b = -V_g \nabla P \)
  drift slowly with a velocity proportional to \( \mathbf{F}_b \)
  dissolve due to diffusion

The structure formation observed in the experiment may be interpreted as an interaction of the sound field and the bubble concentration. Due to (primary) Bjerknes forces [5, 6] the bubbles move to specific locations of the acoustic field. This motion, however, changes the distribution of microbubbles, which by itself has a strong influence on the sound field due to the dependence of the speed of sound on the bubble concentration. Therefore the dynamics of the amplitude of the sound field is given by a nonlinear Schrödinger equation where the potential is replaced by the concentration of bubbles. This Schrödinger equation is coupled with another nonlinear partial differential equation that describes the spatio-temporal evolution of the bubble concentration. For details of the model and its derivation see ref. [7]. In dimensionless form the set of two coupled partial differential equations for the (complex) amplitude \( w \) of the sound field and the concentration \( n \) of microbubbles may be written as

\[
\frac{\partial w}{\partial t} = i \frac{\partial^2 w}{\partial x^2} + nw, \quad (1)
\]

\[
\frac{\partial n}{\partial t} = \gamma \frac{\partial}{\partial y} \left[ n \frac{\partial}{\partial y} (w^* ) \right] - \frac{n - |w|^2}{\tau}, \quad (2)
\]

The parameter \( \tau \) is the (dimensionless) time of dissolution of microbubbles and \( \gamma \) denotes the (dimensionless) characteristic parameter of the Bjerknes forces. The boundary condition for the amplitude \( w \)

\[
\frac{\partial w}{\partial y}(0, t) = \frac{\partial w}{\partial y}(1, t) = 0, \quad (3)
\]

Fig. 3. Plane wave in a channel.
describes the reflection at the lateral walls of the channel. Substituting this condition in (2) yields an ordinary differential equation for the evolution of the concentration \( n \) of bubbles at the boundaries \( y = 0 \) and \( y = 1 \) that can be written in the following form:

\[
\frac{dn}{dt} = -\left[ \frac{1}{\tau} + 2\gamma \text{Im} \left( \frac{2w}{\partial_t} \right) + 2\gamma |w|^2 n \right] n + \frac{|w|^2}{\tau}, \quad (y = 0, 1).
\]  (4)

Solutions of (4) give the time-dependent boundary conditions for equation (2).

2.2. Stability of uniform solutions

It is easy to verify that a uniform solution \( w = A \exp(\imath \theta) \) of the coupled system of partial differential equations (1) and (2) with \( A = A_0 = \text{const.}, n = A_0^2 \) and \( \theta = \frac{\pi}{2} + \theta_0 \) exists. The evolution of a small periodic perturbation with wavelength \( L = 2\pi/K \) is governed by the linearized equations. A stability analysis results in the following stability criterion:

\[
\frac{2A_0^2}{1 + 2\gamma A_0^2} = K_1, \quad K_2 < K_1 < \frac{1}{\tau \gamma A_0^2}
\]  (5)
i.e., there exists a long wavelength instability given by the threshold \( K_1 \) and a short wavelength instability with threshold \( K_2 \).

The threshold \( K_1 \) for the long wavelength instability attains its maximum value

\[
K_1 = \frac{1}{\sqrt{2\gamma}}
\]  (6)

for the critical amplitude

\[
A_0 = \frac{1}{\sqrt{2\gamma}} 0
\]  (7)

Homogeneous distributions of microbubbles and concentrations with long wavelength \( L > L_1 = 2\pi/K_1 \) are therefore not stable. This instability may be interpreted as the reason for the occurrence of localized structures (bubble clusters) as observed in the numerical simulations.

2.3. Integrals of motion

It can be shown that the energy of the sound field

\[
E = \int_0^1 |w(y, t)|^2 \, dy
\]  (8)
is constant and depends only on the initial distribution of the amplitude \( w = w(y, 0) \). The temporal evolution of the total number of bubbles

\[
M = \int_0^1 n(y, t) \, dy
\]  (9)
can be described by the following ordinary differential equation

\[
\dot{M} = \frac{E - M}{\tau}.
\]  (10)

When the initial distribution of bubbles is chosen to be equal to the initial density of energy of the acoustic field, i.e. \( n(y, 0) = w(y, 0)|^2 \), the total number of bubbles \( M \) has to be constant in time. \( M(t) = E(t) = E_0 \). This fact has been used for controlling the accuracy of the numerical method.
Fig. 4(a) and (b).
Fig. 4. Evolution of the bubble concentration $n$ and the amplitude of the sound field $|w|$ for $r = 1$ and $\gamma = 0.02$. (a) Surface plot of the concentration $n$ vs. the space coordinate $y$ and time $t$. (b) Colour-encoded representation of the concentration $n$ vs. the space coordinate $y$ and time $t$. (c) Surface plot of the sound field amplitude $|w|$ vs. the space coordinate $y$ and time $t$. (d) Colour-encoded representation of the sound amplitude $|w|$ vs. the space coordinate $y$ and time $t$.
2.4. **Numerical simulation**

Figure 4 shows a typical result of our numerical simulation based on equations (1) and (2) using finite differences with a spatial resolution of 257 grid points. The size of the time steps was $\Delta t = 0.1 \Delta y^2$ with $\Delta y = 1/257$. The initial distributions of $w$ and $n$ are given by $w_0 = 1 - 0.1 \cos(2\pi y)$ and $n = |w_0|^2$ with $y \in [0, 1]$. In the first stage of the evolution this long wavelength perturbation grows and gives rise to the creation of short waves and localized structures. While at the beginning of the evolution $|w|$ and $n$ are in phase, the locations of the emerging bubble clusters (maxima of $n$) coincide with minima of the sound-field amplitude. As can best be seen in the colour-encoded contour plots in Fig. 4(b) and Fig. 4(d) both distributions, $n$ and $|w|$, undergo several transitions with an increasing number of peaks.

The computation was stopped at $t = 800$, since in physical reality effects now become dominant that are not included in the model (e.g. coalescence of bubbles and secondary Bjerknes forces). Furthermore, due to the strong selfconcentration of microbubbles the peaks of the $n$-distribution become so narrow that the spatial resolution of the grid chosen is probably not sufficient anymore. This is due to the fact that in its present form the model possesses a short wavelength instability and does not contain any physical mechanism (e.g. coalescence) that prevents the concentration $n$ from growing without bounds at minima of the sound-field amplitude. Nevertheless, it is possible to interpret the observed generation of localized structures as the first step towards the formation of branched (fractal) structures (for a discussion see ref. [7]).

The computations have been repeated using different time-steps and grid resolutions. In all cases the first stage of the evolution was reproduced and narrow peaks of the concentration $n$ occurred. Details of the transition phase, however, and the number, size and location of the concentration peaks were found to be very sensitive to modifications of the integration scheme.

3. **PARTICLE APPROACH**

In this approach the bubbles will be treated as particles in a standing acoustic wave. Due to the external sound field from the symmetrical cavitation cylinder every bubble oscillates and experiences primary Bjerknes forces, by which it moves to the centre of the cylinder. In addition to primary Bjerknes forces, so-called secondary Bjerknes forces are acting between the bubbles that can be written in the following form:

$$ F_{ii} = -\frac{\rho}{4\pi} \langle \dot{V}_i \dot{V}_j \rangle \frac{r_i - r_j}{|r_i - r_j|^3}, \tag{11} $$

where $\rho$ is the density of the liquid, $|r_i - r_j|$ is the distance between the pulsating bubbles and $V_i$ the volume of the $i$th bubble [8, 9]. The nature of the secondary Bjerknes forces (in ideal and real media) is just the acoustic radiation force exerted on the first bubble by wave scattering at the second particle [10, 11]. The magnitude and the sign of these Bjerknes forces depend on the relative phase between the bubble pulsations. Bubbles which are oscillating in phase are attracting each other, and bubbles which are oscillating out of phase are repelling each other [12, 13]. In a first approximation all bubbles are assumed to oscillate in phase, so that they all attract each other. Furthermore the spatio-temporal evolution of the patterns formed by the bubbles is not considered. In the current stage of the numerical investigations, we are also not interested in the exact relationship of the physical parameters to the pattern observed in the experiment, but only in the fundamental underlying mechanism which is relevant for the pattern formation. Therefore a simple
(quasi)deterministic aggregation model—instead of integrating the equations of motion, which can be obtained using equation (11)—is investigated. Using this model the influence of the exact values of the physical parameters is neglected, but it is much easier, from a phenomenological point of view, to study the influence of changes of the force on the pattern formation process. In the following the interest is focused on the question, whether the fractal structures (streamers), observable in the experiment, can be explained by the $1/r^2$ dependence of the secondary Bjerknes forces. To make this dependence more dominant, also the primary Bjerknes forces are neglected in the following. By using this simple model, it was possible to carry out simulations with a large number of particles. For studying the influence of the secondary Bjerknes forces on the structure formation, particle aggregation due to the simple force law

$$F_i(r_{N+1}) = \frac{r_{N+1} - r_i}{r_{N+1} - r_i^{1/\alpha}},$$

is investigated that gives the force between the $i$th bubble and the ‘new’ bubble with number $N + 1$ that has not yet aggregated. This model is introduced and discussed in more detail in [14]. The simulations are done as follows: A seed particle is placed in the middle of a circle. Then a particle (bubble) is randomly placed on a circle around the seed particle (cavitation cylinder wall) and moves according to the law (12) until it gets into contact with the already formed cluster. After that a new particle is launched on the circle. A picture of a typical simulation with this model is shown in Fig. 5. In this figure a cluster consisting of 10,000 particles is plotted. The particles were created uniformly distributed on a circle around a seed particle. It can be seen clearly that the result from the numerical simulation gives a much denser structure in comparison to the experiment. Therefore as a first modification of the model, the bubble–bubble interaction force was changed. It is well known, that there exist forces, at least for rigid particles on a short-range scale, with interaction-force exponents different from 2, e.g. König forces, which are $\propto 1/r^2$ [10, 11]. To study the influence of the force exponent $\alpha$ on the structure formation, simulations

Fig. 5. Bubble cluster computed using the aggregation model with $\alpha = 2$. The axes are labelled in units of the bubble diameter.
were done with different exponents $\alpha$, according to a generalized formulation of equation (12):

$$F_i(r_{N+1}) = \frac{r_{N+1} - r_i}{|r_{N+1} - r_i|^{1+\alpha}}.$$  \hspace{1cm} (13)

The result of a simulation, carried out in the same way as described above, for a force-exponent $\alpha = 4$ is shown in Fig. 6(a). Obviously, the pattern shows a much thinner structure than the simulation with $\alpha = 2$ and the picture is in much better agreement with the structures observable in the experiment (compare with Fig. 2). Just to study the

Fig. 6. (a) Bubble cluster computed using the aggregation model with $\alpha = 4$. (b) Bubble cluster computed using the aggregation model with $\alpha = 6$. 
influence of $\alpha$ on the resulting pattern. Fig. 6(b) shows a picture obtained from a simulation with $\alpha = 6$. Although these forces cannot be motivated for the bubble–bubble interaction from physical principles at the moment, it is interesting to see that the pattern obtained from a simulation of equation (13) with $\alpha = 6$ fits best to the structures observable in the experiment. It is obvious that the fractal dimension of the aggregates decreases with increasing exponent $\alpha$ [14, 15] varying from $d \approx 1.8$ for $\alpha = 2$ to $d \approx 1.3$ for $\alpha = 6$. Fractal dimension of the experimental pattern are not yet available due to limitations of our holographic digital processing device. However, visual inspection shows that the pattern for $\alpha = 6$ is most similar to the experimental one.

This fact can be interpreted in two different ways. It could mean that the bubble–bubble interaction forces are much less relevant for the structure formation process than one might expect and that they are only important on very small distances. The mechanisms for the large-scale structures must then be of different nature. It could also mean, that the bubble-creation process (the initial conditions and the boundary conditions) is of great importance for the resulting pattern. A very important difference in contrast to the experimental situation is the uniformly distributed creation of the particles on the circle (cavitation cylinder wall). In the experiment, however, one can observe that the particles are mainly created only in specific regions on the cylinder. Introducing this effect in the numerical model leads to much thinner structures. Figure 7 shows the result of a simulation for $\alpha = 2$ where the bubbles are created on a circle with a radius of 250 bubble diameters, using a nonuniform probability function with local maxima at $\phi_1 = 20^\circ$, $\phi_2 = 75^\circ$, $\phi_3 = 130^\circ$, $\phi_4 = 210^\circ$, $\phi_5 = 260^\circ$ and variances of 10 bubble diameters.

4. CONCLUSION

In this paper two approaches for modelling structure formation processes in cavitation bubble fields are discussed. The first (continuous) model takes into account the interaction of the external sound field and the concentration of bubbles. Linear stability analysis and

![Fig. 7. Bubble cluster computed using the aggregation model with $\alpha = 2$. The bubbles are generated on a circle according to a nonuniform probability distribution with five local maxima.](image)
numerical simulations show that long-wavelength perturbations of uniform distributions of microbubbles are unstable, and give rise to a complex spatio-temporal evolution yielding strongly localized bubble clusters. When this happens the limits of the model in its present form are reached, because physical phenomena become dominant that are not yet included. In particular, bubble–bubble interactions due to secondary Bjerknes forces occur that form the basis of the second model where the bubbles are considered as attracting particles. Simulations assuming different power laws for the secondary Bjerknes forces yield fractal patterns of bubbles similar to the experimental results.

Both models investigated capture only some of the underlying physical processes. The continuous approach, for example, does not include the influence of secondary Bjerknes forces and bubble coalescence. In some sense complementary to that, in the particle model primary Bjerknes forces and the dependence of the sound wave on the bubble concentration are not taken into account. Therefore, in future work both models should be generalized and perhaps be combined in order to obtain a satisfying description of the experimental observations.

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