Inviscid dynamics of a wet foam drop with monodisperse bubble size distribution

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Motivated by recent experiments involving the acoustic levitation of foam drops, we develop a model for nonlinear oscillations of a spherical drop composed of monodisperse aqueous foam with void fraction below 0.1. The model conceptually divides a foam drop into many cells, each cell consisting of a spherical volume of liquid with a bubble at its center. By treating the liquid as incompressible and inviscid, a nonlinear equation is obtained for bubble motion due to a pressure applied at the outer radius of the liquid sphere. Upon linearizing this equation and connecting the cells at their outer radii, a wave equation is obtained with a dispersion relation for the sound waves in a bubbly liquid. For the spherical drop, this equation is solved by a normal mode expansion that yields the natural frequencies as functions of standard foam parameters. Numerical examples illustrate how the analysis may be used to extract foam parameters, such as void fraction and bubble radius, from the experimentally measured natural frequencies of a foam drop. © 2002 American Institute of Physics. DOI: 10.1063/1.1475315

I. INTRODUCTION

Foams and froths are ubiquitous in nature and industry. They are the signature of vigorous, gas-entraining mixing processes in liquids. A minimalistic conception of a foam would consist of a gas confined as bubbles within a liquid host. The aqueous foams considered here are composed of surfactant-bearing water and air bubbles. A comprehensive review of foam theories and applications can be found in the textbooks by Edwards et al. and Exerowa and Kruglyakov, and in the article by Kraynik.

Theoretical treatments of the unique rheology of foams go back at least to Mallock, who was motivated to explain the common observation that “A tumbler containing a frothy liquid gives a dull sound when struck.” Mallock showed that the sound speed for intermediate void fractions was actually lower than its value for either the wet limit, $c_{\text{water}} = 1500 \text{ m/s}$, or the dry limit, $c_{\text{air}} = 340 \text{ m/s}$. This result has been borne out by a century of subsequent work on bubbly liquids and, as seen later, leads to key insights into the free vibrations of foam drops.

The present work is motivated by the continuing need to measure, understand, model, and eventually predict foam mechanics and rheology for wet or dry foams. To this end, two of the present authors recently described a noncontact technique in which small samples of an aqueous foam were acoustically levitated and excited into resonance by modulating the levitation field. The samples were approximately spherical in shape, so will henceforth be referred to as “drops.” Our technique utilizes acoustic levitation to provide both noncontact positioning and static and dynamic excitation of foam drops. By measuring the quadrupole natural frequency of a 3.8 mm radius foam drop to be 63 Hz, we inferred a shear modulus in the range of 73–78 Pa for a relatively dry foam. This value compared favorably with experimentally determined moduli utilizing more traditional contact-based techniques.

The primary advantages of this acoustic levitation technique are its elimination of the requirement for sample contact containment, its ability to test foams of arbitrary gas volume fraction, and its ability to excite both shear and dilatational motion. The technique relies on an accurate physical model for the dynamic response of spheroidal foam drops in order to infer foam properties from experimental measurements. The heart of such a model is a theoretical description of the foam dynamics.

Our earlier work modeled the foam as an effective solid elastic medium. While successful in describing the small-amplitude oscillations of a dry foam with a fixed lattice of bubbles, such a material description has certain disadvantages. First, a foam is only solid-like for high gas vol-
ume fractions and small-amplitude motion. Second, such an effective elastic medium theory only implicitly incorporates the effect of varying gas volume fraction via the effective density of the medium, and it cannot capture the physics of a bubbly mixture. Finally, the gas volume fraction is a dynamic quantity, and during dilatational motion of significant amplitude it cannot be treated as a material constant. We thus need a model capable of describing wet foams.

The subject of the present paper is a theoretical investigation of foam dynamics in the wet limit; such foams are often referred to as “bubbly liquids,” though we are concerned with void fractions that are orders of magnitude greater than the acoustical oceanography community deals with. The model conceptually divides a foam drop into many cells, each cell consisting of a spherical volume of liquid with a bubble at its center. This division was suggested by Wood\textsuperscript{6} for the purpose of analyzing dissipation in bubbly media, and widely used for many problems of multiphase fluid dynamics by Nigmatulin.\textsuperscript{8} The dimensions of the bubble and liquid layer are chosen such that the mass of gas and liquid in the drop are identical to that in the cells. By linearizing the analysis, we find the natural frequencies of the monopole and multipole modes of a drop. We begin in the next section with an argument for the relevant physics that must be included in the model.

II. FOAM MECHANICS, RHEOLOGY, AND DROP DYNAMICS

One of the most important characteristic parameters of a foam is its gas volume fraction $\alpha_g$, or more commonly the “void fraction.” A foam’s thermodynamic, mechanical, acoustical, and rheological properties are sensitive functions of the void fraction. Three regimes of foam morphology are typically identified. A “wet foam” (approximately $0 < \alpha_g < 0.5$) is essentially a bubbly liquid. The individual bubbles are free to move about within the liquid. Wet foams cannot support shearing motion, except at the surfaces of the individual bubbles. A “transitional” or “critical foam” (approximately $0.5 < \alpha_g < 0.7$) is comprised of bubbles whose dynamics are strongly interacting, and whose surfaces may be in mechanical contact with each other. This regime may be usefully thought of as a phase transition between a liquid- and solid-like state. A critical void fraction marks the point at which a foam begins to possess solid-like properties, such as shear wave propagation and yield stress. The critical void fraction for three-dimensional foams is approximately 0.67, which corresponds to random close packing of bubbles. Finally, a “dry foam” is the commonly encountered state in which the bubbles, at least for low to moderate straining rates, have a fixed position in a lattice. Such foams behave as viscoelastic solids for sufficiently small straining rates. However, a dry foam may flow as a liquid when strained beyond a critical point.

Theoretical investigations of rheological properties begin with Derjaguin,\textsuperscript{20} who derived an expression for the shear modulus of an idealized dry foam and showed that it was proportional to the foam capillary pressure. Subsequent theoretical and numerical work has concentrated primarily on two-dimensional geometric models limited to dry foams. The reader is referred to Refs. 21–26 for examples of groundbreaking work in this area and to Ref. 27 for a comprehensive review. Several theoretical models have addressed the unique rheological dependence on void fraction. Bolton and Weaire\textsuperscript{28} introduced a two-dimensional model that predicts the vanishing of the shear modulus at a critical void fraction. This model is strictly valid in the dry limit. In contrast, a static but bubble-based two-dimensional molecular dynamics simulation\textsuperscript{29} captures the transition features using wet-limit assumptions. The model assumes spherical bubbles that resist deformation because of their Laplace pressure. Among other things, the model predicts the vanishing of the shear modulus, but the scaling behavior near the transition is different than that found for dry two-dimensional models.

It is interesting to qualitatively consider the dynamics of foam drops in the limiting cases of wet, critical, and dry. We consider a foam drop surrounded by a gas to simplify the situation. To engage in even a brief discussion, we must draw a distinction between the monopole (or “breathing”) mode and the multipole (or “higher-order shape”) modes, since such motions are qualitatively different. First we consider monopole oscillations. The restoring force for perturbations from the drop’s equilibrium volume is provided by the internal pressure of the individual bubbles within the drop, which will expand and contract when the drop volume is externally forced.

Surface tension plays a small role at the macroscopic level, since the Laplace pressure $2\sigma/R_0$ for a drop of radius $R_0$ is much smaller than the ambient or atmospheric pressure. However, surface tension will have an increasingly important effect as the mean radius $a_0$ of the bubble size distribution approaches 1 $\mu$m, for then the bubble’s Laplace pressure $2\sigma/a_0$ is on the order of the ambient pressure. The mass is that of the liquid between the bubbles. Dissipation is provided by the bulk fluid motion, the surface fluid motions at the drop surface and at the individual bubble surfaces, and also by heat transfer and acoustic radiation of the individual oscillating bubbles.\textsuperscript{30} For critical and dry foam drops, a primary difference is that surface tension becomes more important as a restoring force because of the many thin-film fluid connections that form inside the drop. The mass continues to decrease as the void fraction increases. It is difficult to make any general statement about the effect on dissipation due to increasing void fraction, except to say that dissipative effects increase relative to inertial effects.

For multipole oscillations of wet foam drops, the restoring force for perturbations from the drop’s equilibrium shape is surface tension acting at the drop and bubble interfaces. Since surface-active agents are present, a local Marangoni restoring force due to gradients in surface tension also contributes. The effective mass is once again the mass of the liquid component. The dissipation is more strongly affected by the surface terms at the drop interface than for the monopole case, and the thermal and acoustic bubble dissipation terms are negligible. For shape oscillations of critical and dry foam drops, the internal thin film fluid connections add stiffness to the drop and allow the possibility of torsional multi-
pole modes. As for the monopole case, the mass is decreasing, and dissipation is again ambiguous as the void fraction increases.

The discussion above has implicitly assumed that the internal pressure of the mixture inside the drop is uniform (except for the Laplace pressure contribution to the interior bubble pressure). This will hold true until either the wavelength of incident sound is short compared to the drop radius \( R \) (which will never occur during standing wave acoustic levitation), or the velocity of the drop interface, \( \dot{R} \), approaches the speed of sound in the mixture (which is also unlikely but not impossible since the sound speed in the foam will be quite low), or if the wavelength of sound in the foam is much less than \( R \) (possible again because of the low sound speed in the bubbly mixture). Uniform mixture pressure implies that all bubbles oscillate essentially in phase unless there are wide disparities in the bubble size distribution, or nonlinear effects dominate.

Thus we turn our attention to a more dynamic and bubble-based description of a foam, and in so doing we explicitly incorporate the fact that we utilize acoustic levitation of bounded foam drops in our experiments. We wish to improve upon existing models by considering three-dimensional cases, and by explicitly accounting for the finite volume of the foam drop. We begin here by introducing a three-dimensional model for the free vibration of spheroidal foam drops in the wet (or bubbly liquid) limit.

III. BUBBLE-IN-CELL ANALYSIS OF A WET FOAM SAMPLE

In this section, we present an analysis of a foam sample that is based on the bubble-in-cell model. This model conceptually divides the foam sample into cells, with each cell consisting of a liquid sphere with a gas bubble in its center. The geometry is shown in Fig. 1. The liquid is analyzed first, under the assumptions that it is incompressible and inviscid. By starting with the momentum equation and invoking conservation of mass, one arrives at a nonlinear equation that relates the pressure differential between the inner and outer radii to the motion of the bubble wall. An analysis of the gas in the bubble assumes that the pressure is spatially uniform and is governed by a polytropic gas law, resulting in a relationship between the bubble radius and the liquid pressure at the bubble wall. Linearizing and combining these results for the liquid layer and gas bubble results in a wave equation for the foam sample.

A. Dynamics of one cell

To avoid confusion with the standard notation for velocity potentials (note that several authors use \( \phi \) to denote the void fraction), let \( \alpha_l \) and \( \alpha_g \) be the fractional volume concentration, and \( \rho_l \) and \( \rho_g \) the density of the liquid and the gas, respectively. Then the density \( \rho \) of the two-phase mixture is given by

\[
\rho = \alpha_l \rho_l + \alpha_g \rho_g,
\]

where \( \alpha_l + \alpha_g = 1 \). Note that the gas volume fraction depends on the instantaneous bubble radius,

\[
\alpha_g = \frac{4}{3} \pi a^3 n,
\]

where \( a \) is the instantaneous bubble radius and \( n \) is the number of bubbles per unit volume of the mixture. The radius \( A \) of the liquid sphere in one cell and the bubble radius are related to the void fraction at any instant by

\[
a = \frac{A}{\alpha_g^{1/3}}.
\]

Note that \( A, a, \) and \( \alpha_g \) are variable in time. Typically, \( a < A \ll R \), where \( R \) is the radius of the drop.

Conservation of mass and the incompressibility of the liquid require that the radial mass flow is independent of radius,

\[
v = \frac{\dot{a} a^2}{r^2}, \quad \text{where } a \leq r \leq A,
\]

and \( v \) is the radial velocity of the liquid at \( r \). The mass and bubble densities are related by requiring that the total mass of each cell must not change in time. The initial volume of one cell is \( V_0 = 1/n_0 \), so the initial mass is \( M_0 = \rho_0/n_0 \), where \( \rho_0 = \alpha_{g0} \rho_g + \alpha_{l0} \rho_l \). Requiring that the mass of one cell remains constant gives

\[
\frac{\rho}{n} = \frac{\rho_0}{n_0}.
\]

The dynamics of the liquid layer are analyzed by assuming spherical symmetry and by writing the momentum equation for an incompressible and inviscid fluid,

\[
\frac{1}{\rho_l} \frac{\partial p}{\partial r} = -v \frac{\partial v}{\partial r}.
\]

Integrating this equation from \( r = a \) to \( r \to \infty \) and invoking (4) yields

\[
p = p_a - \rho_0 \left[ a \ddot{a} + \frac{3}{2} \dot{a}^2 - \frac{a}{r} (a \ddot{a} + 2 \dot{a}^2) + \left( \frac{a}{r} \right)^4 \frac{d^2}{2} \right].
\]
where each overdot denotes a time derivative, \( r \) is the radial coordinate with the origin at the bubble center, \( p \) is the liquid pressure at any radius, and \( p_a \) is the liquid pressure evaluated at the bubble wall.

Evaluating (7) at the outer radius of the liquid layer, \( r = A \), and invoking (3) results in

\[
\frac{p_a - p_A}{\rho_l} = (1 - \alpha_g^{1/3})a\ddot{a} + \frac{3}{2} \left( 1 - \frac{4}{3} \alpha_g^{1/3} + \frac{1}{3} \alpha_g^{4/3} \right) \dot{a}^2.
\]  

This equation bears some resemblance to the Rayleigh–Plesset equation that governs the motion of a single bubble in an infinite fluid,

\[
\frac{p_a - p_\infty}{\rho_l} = a\ddot{a} + \frac{3}{2} \dot{a}^2,
\]  

where \( p_\infty \) is the liquid pressure far from the bubble. In the limit of vanishing void fraction, \( \alpha_g \to 0 \), (8) reduces to (9) as one would expect.

Now we turn our attention to the dynamics of a bubble at the center of one cell and assume that the gas pressure inside the bubble, \( p_g \), is spatially uniform. Continuity of momentum requires that the gas pressure differs from the liquid pressure at the bubble wall by a surface tension term, according to

\[
p_g = p_a + \frac{2\sigma}{a},
\]  

where \( \sigma \) is the surface tension coefficient. Furthermore, the gas pressure is assumed to be governed by a polytropic gas law of the standard form

\[
p_g = \left( \frac{p_0 + 2\sigma}{a_0} \right) \left( \frac{a}{a_0} \right)^{-3\kappa},
\]  

where \( a_0 \) is the mechanical equilibrium radius of the bubble, \( \kappa \) is the polytropic exponent (\( \kappa = 1 \) for isothermal oscillations and \( \kappa = \gamma_g \) for adiabatic oscillations, where \( \gamma_g \) denotes the gas adiabatic exponent). Combining (8)–(11) gives the following relationship between the pressure at the edge of the cell and the bubble radius:

\[
p_A = \left( \frac{p_0 + 2\sigma}{a_0} \right) \left( \frac{a}{a_0} \right)^{-3\kappa} - \frac{2\sigma}{a} - \rho_l (1 - \alpha_g^{1/3}) a\ddot{a} + \frac{3}{2} \left( 1 - \frac{4}{3} \alpha_g^{1/3} + \frac{1}{3} \alpha_g^{4/3} \right) \dot{a}^2.
\]

### B. Dynamics of a cluster of cells

Equation (12) can be integrated to find the bubble radius given the time-dependent pressure at \( r = A \) and initial conditions. However, our goal here is to analyze the coupled dynamics of many connected cells. Therefore, we conceptually connect the cells at points on their outer radii and replace the assemblage with an “equivalent” fluid whose dynamics approximate those of the foam. This arrangement is shown schematically in Fig. 2. We assume all the bubbles have equilibrium radius \( a_0 \). The fluid is required to be mass-conserving, so that

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]

where \( \rho \) is given in (1). The fluid also satisfies Euler’s equation,

\[
\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0,
\]

where \( D(\cdot)/Dt \) is the substantial derivative.

We linearize our equations by assuming that time-dependent quantities only vary slightly from their equilibrium values. Specifically, we write \( \rho = \rho_0 + \rho' \), \( n = n_0 + n' \), \( p_L = p_0 + p' \), \( \mathbf{v} = \mathbf{v}' \), \( \alpha_g = \alpha_{g0} + \alpha_g' \), and \( a = a_0 + a' \). The primed quantities are assumed small, such that any product of primed quantities may be neglected. When these assumptions are introduced in (12), we arrive at the linearized equation

\[
\rho' = - \left[ \left( \frac{p_0 + 2\sigma}{a_0} \right) \left( \frac{a}{a_0} \right)^{-3\kappa} - \frac{2\sigma}{a_0} \right] a' - \rho_l (1 - \alpha_g^{1/3}) a_0 a\ddot{a}'.
\]

Linearizing (5) gives

\[
n' = \frac{n_0}{\rho_0} \rho'.
\]

Similarly linearizing the density in (1) and using (16) gives

\[
\rho' = -4\pi a_0^2 n_0 \rho_0 \left( \frac{\rho_l - \rho_g}{\rho_l} \right) a'.
\]

Since the mass density of the liquid is typically much larger than that of the gas in the bubbles and our interest is in low void fractions, we shall proceed with the approximation

\[
\frac{\rho_l - \rho_g}{\rho_l} \approx 1.
\]

This approximation simplifies (17) to

\[
\rho' = -4\pi a_0^2 n_0 \rho_0 a'.
\]

We also linearize (13) and (14) and use (18) to get

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0,
\]

\[
\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0,
\]

where \( \rho_0 = \rho_l (1 - \alpha_{g0}) \).
Introducing the velocity potential gives

\[ v = \Phi, \]

and

\[ \ddot{v} + k^2 v = 0. \]

Substituting this result in the Laplacian of Eq. (2), we see that this pressure disturbance is equated to a pressure disturbance in the equivalent fluid. In addition, the outer radii of adjacent cells are assumed to co-incide at all times. As indicated in the analysis below, the assemblage of connected cells is replaced by an acoustic fluid that is equivalent in the sense that it has the same pressure–density relations.

The derivation of the linearized equation for the velocity potential in the equivalent fluid begins by combining (15) and (19) to yield a pressure–density relation for the equivalent fluid,

\[ p' = c_b^2 \rho' + \left( \frac{c_b}{\omega_b} \right)^2 \ddot{\rho}', \]

where \( c_b \) represents the sound speed in the foam,

\[ c_b^2 = \frac{3 k p_0 + (3 k - 1) \frac{2 \sigma}{a_0}}{3 \alpha g_0 \rho_0}, \]

\( \omega_b \) is the natural frequency of a bubble in the foam,

\[ \omega_b^2 = \omega_{sb}^2 \left( 1 - \frac{3}{\alpha g_0} \right), \]

and \( \omega_{sb} \) is the natural frequency of a single bubble in an infinite liquid,

\[ \omega_{sb}^2 = \frac{3 k p_0 + (3 k - 1) \frac{2 \sigma}{a_0}}{\rho_0 a_0^2}. \]

Next, Eqs. (20)–(22) are combined to yield a wave equation for the velocity potential. Combining the time derivative of Eq. (20) with the divergence of Eq. (21) results in

\[ \frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 p'. \]

Substituting this result in the Laplacian of Eq. (22) and introducing the velocity potential gives

\[ \frac{\partial^2 \varphi}{\partial t^2} - c_b^2 \nabla^2 \left( \varphi + \frac{1}{\omega_b^2} \frac{\partial^2 \varphi}{\partial t^2} \right) = 0. \]

IV. NORMAL MODES OF A SPHERICAL FOAM DROP

Now consider a foam sample consisting of a spherical liquid drop of radius \( R \) with \( N \) spherical gas bubbles dispersed inside. The drop is surrounded by the same gas that is in the bubbles and that gas is held at a constant pressure \( P_\infty \). To derive formulas for the eigenfrequencies of such a foam sample, let the velocity potential inside the foam drop be time harmonic,

\[ \varphi = \Re\{ \Phi \exp(i \omega t) \}, \]

where \( \omega \) is the unknown natural frequency. The substitution of (28) into (27) results in the Helmholtz equation for the complex amplitude of the velocity potential,

\[ \nabla^2 \Phi + k^2 \Phi = 0, \]

where

\[ k^2 = \frac{\omega^2}{c_b^2 \left( 1 - \frac{2 \sigma}{a_0} \right)} - \frac{\omega^2}{\omega_{sb}^2 - \omega^2}. \]

Here, let us pause to compare (30) to the classic result of a wavenumber for a bubbly mixture.\(^3\) Rewriting (30) as

\[ k^2 = 4 \pi a_0 \left( \frac{\omega^2}{\omega_{sb}^2 - \omega^2} \right). \]

These expressions differ in two ways. First, (31) has \( \omega_b^2 \) in the denominator while (32) has \( \omega_{sb}^2 \). Since \( \omega_b \) is the resonance of a bubble in the foam, we expect the wave number to become infinite when \( \omega = \omega_b \), and therefore (31) is more accurate. Second, (31) has a geometric factor involving \( \alpha g_0 \). The numerator of \( 1 - \alpha g_0 \) would also appear in the classical result if the assumption of \( \rho = \rho_0 \) had not been used in its derivation. The denominator of \( 1 - \alpha g_0 \) is a direct consequence of the finite liquid inertia seen by the bubble in the bubble-in-cell model. Instead, the classical result uses a radiation mass of \( 4 \pi a_0 \rho_0 \) that measures the inertia of an infinite fluid as seen by the bubble wall. Despite these differences, the results agree when \( \alpha g_0 \ll 1 \).

The normal mode solution of (29) is

\[ \Phi = \sum_{n=0}^{\infty} B_n j_n(kr) P_n(\cos \theta), \]

where the \( j_n(kr) \) are spherical Bessel functions, the \( P_n(\cos \theta) \) are Legendre polynomials, and the \( B_n \) are the modal amplitudes. In order to compare with experiment, we need to determine the instantaneous radius of the drop. The velocity of the outer radius is found by evaluating \( \mathbf{v} = \nabla \Phi \) at \( r = R \). Integrating to get displacement and adding the mechanical equilibrium radius \( R_0 \) gives an equation for the instantaneous radius,

\[ R = R_0 + \Re \left( \sum_{n=0}^{\infty} b_n P_n(\cos \theta) \exp(i \omega t) \right), \]

where the coefficients \( b_n \) are related to those of the velocity potential by

\[ b_n = \frac{k j_n(k R_0)}{i \omega} B_n. \]
\[ p_0 - p_0 \frac{\partial \varphi}{\partial t} \bigg|_{r=R_0} = \varphi + \rho_\infty. \]  

(36)

The Laplace pressure is given as

\[ p_\sigma = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right), \]  

(37)

where \( R_1 \) and \( R_2 \) are the principal local radii of curvature. Evaluating these radii using the radius in (34) gives\(^{32}\)

\[ p_\sigma = \frac{2 \sigma}{R_0} \]

\[ + \Re \left( \sigma \sum_{n=0}^\infty \frac{(n-1)(n+2)}{R_0^2} b_n P_n(\cos \theta) \exp(\iota \omega t) \right). \]  

(38)

Evaluating (36) and (38) at equilibrium relates the exterior and interior equilibrium pressures by

\[ p_0 = \frac{2 \sigma}{R_0} + p_\infty. \]  

(39)

Substituting (28), (33), (35), (38), and (39) into (36) gives

\[ \Re \left( \rho_0 \sum_{n=0}^\infty \frac{\omega^2 j_n(kR_0)}{k} \frac{j_n(kR_0)}{j_n(kR_0)} b_n P_n(\cos \theta) \exp(\iota \omega t) \right) \]

\[ = \Re \left( \frac{\sigma}{R_0} \sum_{n=0}^\infty \frac{(n-1)(n+2)}{R_0^2} b_n P_n(\cos \theta) \exp(\iota \omega t) \right). \]  

(40)

Equating the \( n \)th coefficients in the series yields an implicit equation for the natural frequency of the drop,

\[ \omega_n^2 = \frac{k \sigma(n-1)(n+2)}{\rho_0 R_0^2} \frac{j_n(kR_0)}{j_n(kR_0)}, \]  

(41)

where \( k \) is evaluated by setting \( \omega = \omega_n \) in (30). This makes the equation nonlinear in \( \omega_n \). In the following sections, we introduce additional assumptions that allow the derivation of explicit forms for the breathing mode \((n=0)\) and shape mode \((n>0)\) natural frequencies.

### A. Breathing mode natural frequency

The monopole natural frequency is found by setting \( n = 0 \) in (41), which yields

\[ \omega_0^2 = -\frac{2 \sigma k}{\rho_0 R_0^2} j_0(kR_0). \]  

(42)

Using an explicit expression for the spherical Bessel function of zeroth order,\(^{33}\)

\[ j_0(kR_0) = \frac{\sin(kR_0)}{kR_0}, \]  

(43)

(42) is rewritten in dimensionless form as

\[ \frac{z^2}{1+bz^2} \sin(\frac{z}{z}) + s \left( \cos(\frac{z}{z}) - \sin(\frac{z}{z}) \right) = 0, \]  

(44)

where the nondimensional variables are

\[ z = kR_0, \]

\[ b = \frac{c_b^2}{\omega_0 R_0^2}, \]

\[ s = \frac{2 \sigma}{\rho_0 R_0 c_b}. \]  

(45)\quad (46)\quad (47)

The effects of surface tension are removed by setting \( s = 0 \). If this is done, the solution to (44) is simply

\[ z = m \pi, \]  

where \( m = 1, 2, \ldots \). \quad (48)

When \( s \) is small, we may seek a correction to \( z \) of the form

\[ z = m \pi + \varepsilon. \]  

(49)

The substitution of (49) into (44) leads to the solution

\[ z = m \pi \left( 1 - \frac{1+b \pi^2}{\pi^2} s \right), \]  

(50)

which is cast in dimensional form as

\[ \omega_0^2 \left( \frac{R_0}{a_0} \right)^2 = \frac{3 \alpha_{e0}(1-\alpha_{e0})}{(m \pi)^2} \left( 1 + 2s \frac{1+b \pi^2}{\pi^2} \right) + 1 - \alpha_{e0}^2. \]  

(51)

It is interesting to note that the effect of including surface tension is to lower the natural frequency. It is also interesting to note that it is possible for the foam drop to possess a natural frequency greater than that for individual bubbles.

In Fig. 3 we plot the breathing mode frequency from (51) as a function of equilibrium void fraction for a variety of bubble sizes. For all bubble sizes, the frequency asymptotes to the single bubble, infinite fluid breathing mode frequency \( f_{sb} \) for a zero void fraction, and increases as the void fraction approaches unity, though the current model will not capture the dominant physics much beyond void fractions \( \alpha_{e0} \) greater than about 0.5. The dependence of the monopole...
frequency on void fraction is strongest for smaller bubbles. For wet foam drops (i.e., $\alpha_{g0} < 0.5$), the strong dependence of the breathing mode frequency on the void fraction would allow the inference of a void fraction from the measured breathing mode frequency given also a measurement of the mean bubble size. If, conversely, the void fraction were known, then a measurement of the breathing mode frequency would yield an estimate of the mean bubble size, especially for bubbles on the order of a micron in radius, where optical techniques begin to fail.

When $R_0 \gg a_0$, the second term in the denominator of (51) may be neglected. If we further neglect the surface tension term by setting $s = 0$, we get the simplified result

$$\omega_n^2 \approx \frac{m \pi}{R_0^2} \frac{\kappa p_0}{\alpha_{g0} \rho_0}.$$  \hspace{1cm} (52)

For $m = 1$, this result is similar in form to the well-known Minnaert formula $^{31}$ for single bubble monopole oscillations,

$$\omega_n^2 = \frac{3 \gamma p_0}{\rho_{l}}$$  \hspace{1cm} \hspace{1cm} (53)

where $\rho_l$ is the density of the liquid external to the bubble. To better see this, (52) may be rewritten for $m = 1$ as

$$\omega_n^2 \approx \frac{3 \kappa p_0}{R_0^2 \rho_{\text{eff}}},$$  \hspace{1cm} (54)

where the effective density is

$$\rho_{\text{eff}} = \frac{3 \alpha_{g0}}{s^2} \rho_0.$$  \hspace{1cm} (55)

These results may be qualitatively compared to previous results for the breathing mode of a compact bubble cloud in water derived by several authors.\textsuperscript{34-37} In those works, the motivation was to explain low-frequency ambient noise in the ocean as due to collective oscillations of clouds of bubbles. The low void fraction limit is the same, but the high void fraction limit is not, since in the present case the high void fraction limit corresponds to the effective mass of the oscillator approaching zero, whereas in the bubble cloud scenario, the high void fraction limit corresponds to a single bubble of radius $R$.

**B. Shape mode natural frequency**

The natural frequency of the shape mode is found by nondimensionalizing (41) to

$$\frac{z}{1 + b z^2} = (n - 1)(n + 1) \frac{s}{2} j'_n(z),$$  \hspace{1cm} (56)

where the dimensionless variables $z$, $b$, and $s$ are defined in (45)–(47).

To find the low-frequency solution of this equation, we assume small $z$ and use the asymptotic expansions for the spherical Bessel function and its derivative,\textsuperscript{33}

$$j_n(z) \approx \frac{z^n}{1 \cdot 3 \cdot 5 \cdots (2n + 1)} \left( 1 - \frac{z^2/2}{2n + 3} \right).$$  \hspace{1cm} (57)

Inserting this approximation into (56) and neglecting powers of $z$ beyond two gives

$$\frac{z}{1 + b z^2} = (n - 1)(n + 2) \frac{s}{2} \left[ n - \frac{1 + n/2}{2n + 3} \frac{z^2}{2} \right].$$  \hspace{1cm} (58)

Introducing dimensional variables, performing a Taylor series about $s = 0$, and neglecting the term $nb$ gives

$$\omega^2 = \omega^2_{L_n} \left( 1 - \frac{(n - 1)(n + 2)(1 + n/2)}{2(2n + 3)R_0} \right) \times \frac{3 \sigma a_{g0}}{3 \kappa p_0 + (3 \kappa - 1)(2 \sigma / \alpha_{g0})},$$  \hspace{1cm} (59)

where

$$\omega^2_{L_n} = \frac{\pi}{3} \frac{(n - 1)(n + 2) \sigma}{\rho_0 R_0^3}.$$  \hspace{1cm} (60)

Here $\omega_{L_n}$ is the Lamb frequency of shape oscillations for an inviscid and incompressible liquid drop\textsuperscript{35} of density $\rho_0$ surrounded by a vacuum. When $n = 2$ and the second term in the brackets is neglected, the drop frequency reduces to the the quadrupole Lamb frequency for a fluid of effective density $\rho_0$.

Figure 4 plots the first few modal frequencies from (61).
as a function of the void fraction for a fixed bubble size. The quadrupole (and higher multipoles) does not show as sensitive a dependence on the void fraction as the breathing mode, which is to be expected since the quadrupole motion is a shearing, not a compressional motion. The increase in frequency with void fraction is a simple result of the decrease of inertia. There is essentially no dependence on mean bubble size for the range of bubble sizes treated in Fig. 3, which means that the correction factor in Eq. (60) is much less than unity for all practical cases.

Note that the modal frequencies described by (61) are those for which the sole restoring force is surface tension acting at the drop interface. This restoring force will compete with the elastic restoring force (again surface tension, but acting at the bubble interfaces as well), for void fractions above the critical void fraction. In Ref. 10, we measured a quadrupole mode frequency of 63 Hz for a foam drop with the same parameters as given in the caption of Fig. 3. In that work we ascribed the restoring force to elasticity, ignoring the contribution of surface tension, resulting in an estimate of 75 Pa for the shear elastic modulus of the foam. Figure 4 shows that assumption to be a good one, since the current model at void fraction of 0.77 underpredicts the quadrupole frequency by a factor of 3. Thus, the strain energy exceeds the free surface energy by approximately an order of magnitude just beyond the critical void fraction.

V. CONCLUSIONS

We have developed a nonlinear model for the oscillations of a wet foam consisting of noninteracting and monodisperse bubbles. By imposing the boundary conditions corresponding to our experimental technique, we have further derived expressions for the normal mode frequencies for spherical foam drops. These results, especially (51) and (61), should be easily verified experimentally using an apparatus similar to that described in Ref. 10.

While we believe these results to be of sufficient scientific interest on their own, our longterm goal is to use this knowledge for rheological purposes. Thus, we note that it appears that the most easily excited and measured mode, the quadrupole, appears to be of limited use in the wet limit, since it exhibits only weak dependence on the void fraction. It may be useful near the transition to elastic behavior near the critical void fraction since our results show that there should be a transition between surface tension dominated behavior, and elastic solid behavior as the void fraction increases. Since dissipative effects due to bubble–bubble interactions will become very important near the critical void fraction, the modeling effort reported here must be viewed as a first step.

The breathing mode is much more sensitive to both the void fraction and the mean bubble size. It is much more difficult to excite directly without simultaneously driving the shape modes, however. Modulation of the ambient pressure, though practically difficult, would accomplish such direct excitation and allow a comparison with the predictions of Eq. (51).

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