RESULTS ON CONVERGENCE OF FOURIER SERIES

(References are from the book *Fourier Analysis: An introduction* by Stein and Shakarchi)

Let $f : [-\pi, \pi] \to \mathbb{C}$ be a Lebesgue integrable function. Then the Fourier coeffs of f are defined by $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$, and the partial sums of the Fourier series of f are $S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}$. Here are the results we have proved about the convergence of $S_N f$ to f, ordered by decresing regularity of f:

- If $f \in C^k(\mathbb{T})$:

  1. If $k \geq 2$, then $|\widehat{f}(n)| = O(\frac{1}{|n|^k})$ as $n \to \infty$, which implies that $S_N f$ converges to f **uniformly** on $\mathbb{T}$ (Chapter 2, Corollary 2.4, and Ex. 10).

     Note: From Riemann-Lebesgue Lemma, we obtain a better decay of the coefficients: If $k \geq 0$, then $|\widehat{f}(n)| = o(\frac{1}{n^k})$ as $n \to \infty$ (Ch.3, Ex.13).
  2. An improvement: For $k \geq 1$, $\|S_N f - f\|_{L^\infty(\mathbb{T})} = O\left(\frac{1}{N^{k-1/2}}\right)$ as $N \to \infty$, and hence $S_N f$ converges to f **uniformly** on $\mathbb{T}$ (Class notes, Ch.3 Sec.2). See also Ch.3, Ex.14 for the case $k = 1$.

- If f is Lipschitz:

  1. $|\widehat{f}(n)| = O(1/|n|)$ (Ch.3, Ex.15).
  2. By Dini's Criterium, $S_N f(x) \to f(x)$ for every $x \in \mathbb{T}$ (Ch.3, Sec.2). Although by Dini's Criterium we cannot prove that the convergence is uniform, that is in fact the case, as seen in Ch.3 Ex.16.
  3. A slight variation of the arguments for f Lipschitz shows that if f is Hölder-α for $0 < \alpha < 1$, then $|\widehat{f}(n)| = O(1/|n|^\alpha)$. And if $1/2 < \alpha < 1$, the Fourier series of f converges uniformly to f (Ch.3 Ex.15, 16).

- If $f \in C(\mathbb{T})$:

  1. If $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$, then $S_N f$ converges to f **uniformly** (Ch.2, Corollary 2.3).
  2. If $\widehat{f}(n) = 0$ for every $n \in \mathbb{Z}$, then $f \equiv 0$ (Ch.2, Corollary 2.2).
  3. **Pointwise convergence:** There exists a continuous function f such that $S_N f(x)$ does not converge to $f(x)$ for some x. This is a consequence of the fact that the Dirichlet kernels are not uniformly bounded in L^1.

     In fact, there exist a dense G_δ subset E of $C(\mathbb{T})$ such that for every $f \in E$, the set $A_f = \{x \in \mathbb{T} : \sup_N S_N(f)(x) = \infty\}$ is a dense G_δ subset of $\mathbb{T}$.

- If $f \in L^2(\mathbb{T})$:

  1. **Norm convergence:** By Hilbert space theory, $\|S_N f - f\|_{L^2(\mathbb{T})} \to 0$ as $N \to \infty$ (Ch.3, Sec.1).
  2. Plancherel's identity holds: $\|f\|^2_{L^2(\mathbb{T})} = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$. This implies that the Fourier transform is an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ (Ch.3, Sec.1).
  3. **Pointwise convergence:** By Carleson Theorem, $S_N f(x)$ converges to $f(x)$ for almost every x. (See discussion in page 5)

- If $f \in L^1(\mathbb{T})$:

  1. If $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$, then $S_N f$ converges uniformly to a continuous function g such that $f = g$ a.e. (Same proof as Ch.2, Corollary 2.3).
  2. **Uniqueness of Fourier series:** If $\widehat{f}(n) = 0$ for every $n \in \mathbb{Z}$, then $f \equiv 0$ a.e., and in particular $f(x_0) = 0$ at any point x_0 where f is continuous (Ch.2, Theorem 2.1).
  3. **Norm convergence:** There exists a function $f \in L^1(\mathbb{T})$ such that $S_N f$ does not converge to f in the L^1 norm. This is because the L^1-norms of the Dirichlet kernels are not uniformly bounded (Ch.2 Prob 2).
  4. **Pointwise convergence:** Kolmogorov constructed an explicit counterexample of an L^1 function whose Fourier series diverges everywhere.
  5. **Dini's criterium and its consequences**:

     (a) At any point x_0 where Dini's criterium holds, $S_N f(x_0) \to f(x_0)$ as $N \to \infty$ (Ch.3, Sec.2).

     (b) *Localization:* If f is identically 0 on an interval (a, b), then $S_N f(x) \to 0$ for every $x \in (a, b)$.

     (c) *Jump discontinuities:* If $f \in L^1(\mathbb{T})$, and at a point x_0 the lateral limits $f(x_0+), f(x_0-), f'(x_0+), f'(x_0-)$ exist, then $S_N f(x_0) \to \dfrac{f(x_0+) + f(x_0-)}{2}$.

     Also, Gibbs phenomenon shows that all the partial sums $S_N f$ overshoot f at the jump point by approximately 9% of the jump at x_0, $(f(x_0+) - f(x_0-))$.

OTHER MODES OF CONVERGENCE OF FOURIER SERIES

- **Cesàro Convergence:**

  Let $\sigma_N f(x) = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} S_N f(x)$ be the Cesàro sums of the Fourier series of f. Then $\sigma_N f(x) = F_N \star f(x)$, where F_N is the N-th Féjer kernel (a good kernel) and hence we have:

  1. If $f \in C(\mathbb{T})$, $\sigma_N f$ converges to f uniformly on $\mathbb{T}$ (Ch.2, Theorem 5.2).
  2. If $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$, then $\lim\limits_{N\to\infty} \|\sigma_N f - f\|_{L^p(\mathbb{T})} = 0$ (Class notes and Hw 1).

- **Abel Convergence:**

  For $0 \leq r < 1$, let $A_r f(x) = \displaystyle\sum_{n=-\infty}^{\infty} r^{|n|}\widehat{f}(n)e^{inx}$ be the Abel sums of the Fourier series of f. Then $A_r f(x) = P_r \star f(x)$, where P_r is the Poisson kernel (a good kernel) and hence we have:

  1. If $f \in C(\mathbb{T})$, $A_r f$ converges to f uniformly on $\mathbb{T}$ (Ch.2, Theorem 5.2).
  2. If $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$, then $\lim\limits_{r\to 1^-} \|A_r f - f\|_{L^p(\mathbb{T})} = 0$ (Class notes and Hw 1).

     Besides results 1 and 2, that follow from the properties of good kernels, for Abel summation we get much more:

  3. If $f \in L^1(\mathbb{T})$ and we define $u(r,\theta) := A_r f(\theta)$, then $u \in C^\infty((0,1)\times(0,2\pi))$, by absolute and uniform convergence properties of power series.
  4. If $f \in L^1(\mathbb{T})$, $\lim\limits_{r\to 1^-} A_r f(x) = f(x)$ a.e., by the weak-$(1,1)$ estimate for the Hardy-Littlewood maximal function.

THE FOURIER TRANSFORM

Let $f : \mathbb{R} \to \mathbb{C}$ be an $L^1(\mathbb{R})$ function. The (continuous) Fourier transform of f is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx.$$

- If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuous, and bounded by $\|f\|_{L^1}$.
- **Riemann-Lebesgue:** If $f \in L^1(\mathbb{R})$, $\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0$.
- By analogy with the summation of Fourier series, the inverse Fourier transform should be given by $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$. But, in general, the Fourier transform of an L^1 function is not in L^1, hence the integral in the inversion formula is not well defined.
- Unlike in the case of Fourier series, where $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$ if $p > 1$, now there is no relation between the spaces $L^p(\mathbb{R})$ for different values of p. Hence the Fourier transform is not (yet) defined for $f \in L^p(\mathbb{R})$, $p \neq 1$.
- Let $\mathcal{S}$ be the class of Schwartz functions ($C^\infty(\mathbb{R})$ functions whose decay at infinity is faster than any polynomial, and whose derivatives also have this decay). Then:
  1. $\mathcal{S}$ is a dense subspace of $L^p(\mathbb{R})$, $1 \leq p < \infty$. In particular, the Fourier transform is well defined for $f \in \mathcal{S}$.
  2. If $f \in \mathcal{S}$, then $\widehat{f} \in \mathcal{S}$ and the inversion formula holds.
  3. **Plancherel:** If $f \in \mathcal{S}$, then $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}$.
- If $f \in L^2(\mathbb{R})$, we use Plancherel's formula and the density of $\mathcal{S}$ in L^2 to define the Fourier transform of f. The Fourier transform thus defined is an isometric isomorphism on $L^2(\mathbb{R})$.
- **Interpolation to $L^p(\mathbb{R})$, $1 < p < 2$:** Since we have

  $$\begin{aligned} \|\widehat{f}\|_{L^\infty} &\leq \|f\|_{L^1}, \\ \|\widehat{f}\|_{L^2} &= \|f\|_{L^2} \end{aligned}$$

  then, for $p \in (1,2)$ and $f \in L^p(\mathbb{R})$, we get, by Riesz-Thorin interpolation,

  $$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

  where $p' = (p-1)/p$ is the conjugate exponent of p. Thus the Fourier transform can be defined for $f \in L^p(\mathbb{R})$, $1 < p < 2$, and in this case $\widehat{f}$ is a function in the dual space $L^{p'}(\mathbb{R})$.
- **The Fourier transform of distributions.** If $f \in L^p(\mathbb{R})$ for $p > 2$, then in general the Fourier transform of f is not a function, but it can be defined in the sense of distributions: The Fourier transform of f is defined as the distribution T such that, for every $\phi \in \mathcal{S}$,

  $$\langle T, \phi \rangle = \int_{-\infty}^{\infty} f(x) \widehat{\phi}(x) dx.$$

Recovering f from $\widehat{f}$.

Given $R \in \mathbb{R}$, $R > 0$, we define S_R (the partial sums of the Fourier transform) by $(S_R f)\widehat{\ }(\xi) = \widehat{f}(\xi)\chi_{[-R,R]}(\xi)$, or equivalently by

$$S_R f(x) = \int_{-R}^{R} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi, \tag{1}$$

and we study in which sense $S_R f$ converges to f as $R \to \infty$.

- If $f \in \mathcal{S}$, $f(x) = \displaystyle\int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$, pointwise.
- If $f \in L^2(\mathbb{R})$, then $\|S_R f - f\|_2 \to 0$ as $R \to \infty$.
- If $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$, then $\|S_R f - f\|_p \to 0$ as $R \to \infty$ if and only if there exists a constant $C_p > 0$, independent on R, such that for every $f \in L^p(\mathbb{R})$

$$\|S_R f\|_p \le C_p \|f\|_p. \tag{2}$$

- For $p = 1$, the estimate (2) does not hold since the L^1-norms of the Dirichlet kernels are not uniformly bounded (Ch.2 Prob 2). Hence in general $S_R f$ does not converge to f in the L^1 norm.
- For $1 < p < \infty$, estimate (2) holds, and thus $\|S_R f - f\|_p \to 0$ as $R \to \infty$. This result is a consequence of the boundedness of the Hilbert transform on L^p, $1 < p < \infty$.
- **Pointwise Convergence:** The pointwise convergence of the partial sums of Fourier series for $f \in L^p$ is one of the hardest results in Analysis. In 1966 L. Carleson proved that if $f \in L^2$, then the partial sums converge to f almost everywhere. (In the discrete case, for all $f \in L^2(\mathbb{T})$, $S_N f(x) \to f(x)$ $a.e.x \in \mathbb{T}$, and in the continuous case, for every $f \in L^2(\mathbb{R})$, $S_R f(x) \to f(x)$ $a.e.x \in \mathbb{R}$). In 1968 Hunt extended this result to L^p, $1 < p < \infty$.

  As mentioned in the first page, for L^1 almost everywhere convergence fails. If we consider Orlicz spaces, the best results so far are:

  Antonov, 1996: For every $f \in L\log_+(L)\log_+\log_+\log_+(L)$, the Fourier series of f converges to f a.e.

  Konyagin, 2000: There exists a function in $L\log_+(L)^{1/2-\epsilon}$ whose Fourier series is everywhere divergent.

- **Other modes of convergence:** There is a version of the Féjer kernel $\mathcal{F}_R$ for the continuous Fourier transform, which is an approximate identity. Thus, if f is uniformly continuous on $\mathbb{R}$, $\mathcal{F}_R \star f$ converges to f uniformly as $R \to \infty$, and if $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$, $\|\mathcal{F}_R \star f - f\|_p \to 0$ (see Ch.5 Ex.9).

APPLICATIONS OF FOURIER ANALYSIS SEEN IN CLASS

1. Solution of various **differential equations** (although convergence was not proved in all cases):

   The wave equation

   - The vibrating string (Ch.1, Sec.1), with its conservation of energy (Ch.3, Ex.10).
   - The general solution of the wave equation in dimensions 1, 2, 3 (Ch.6, Sec.3)

   The Laplace equation

   - The complete solution of the Dirichlet problem $\Delta u = f$ on the disc (Thm 2.5.7), and the pointwise convergence of u to f (using the Hardy-Littlewood maximal function). See Ch.2, Ex.18 for a non-uniqueness result (the function fails to verify one of the conditions of Thm 2.5.7); and Problem 1 in HW2 for the maximum principle.
   - Dirichlet problem for the Laplacian on a rectangle (Ch.1, Prob 1), on a semi-infinite strip (Ch.2, Ex.19), on an annulus (Ch.2, Ex.20), on the upper-half plane (Ch.5, Sec.2)
   - The fundamental radial solution for the Laplacian in $\mathbb{R}^3$ (class notes, Theory of distributions).

   The heat equation

   - The heat equation on the upper-half-plane (Ch.5, Sec.2, Ex 11-12)

2. The Isoperimetric Inequality (Ch.4, Sec.1).
3. Weyl's Equidistribution Theorem (Ch.4, Sec.2).
4. Construction of a continuous, nowhere differentiable function (Ch.4, Sec.3).
5. Poisson's Summation Formula (Ch.5, Sec.3)
6. Heisenberg's Uncertainty Principle (Ch.5, Sec.4)
7. Shannon's Sampling Theorem (Ch.5, Ex.20)

Other applications that we have not seen include: Number theory (Chapter 8), Geometric tomography (reconstruction of images of a body from its lower-dimensional sections or projections), and more...