Thus

$$u(x,t) = \sum_{m=1}^{\infty} A_m \cos mt \sin mx, \tag{8}$$

and note that this series converges absolutely. The solution can also be expressed in terms of traveling waves. In fact

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2}. \tag{9}$$

Here $f(x)$ is defined for all x as follows: first, f is extended to $[-\pi, \pi]$ by making it odd, and then f is extended to the whole real line by making it periodic of period 2π, that is, $f(x + 2\pi k) = f(x)$ for all integers k.

Observe that (8) implies (9) in view of the trigonometric identity

$$\cos v \sin u = \frac{1}{2}[\sin(u+v) + \sin(u-v)].$$

As a final remark, we should note an unsatisfactory aspect of the solution to this problem, which however is in the nature of things. Since the initial data $f(x)$ for the plucked string is not twice continuously differentiable, neither is the function u (given by (9)). Hence u is not truly a solution of the wave equation: while $u(x,t)$ does represent the position of the plucked string, it does not satisfy the partial differential equation we set out to solve! This state of affairs may be understood properly only if we realize that u does solve the equation, but in an appropriate generalized sense. A better understanding of this phenomenon requires ideas relevant to the study of "weak solutions" and the theory of "distributions." These topics we consider only later, in Books III and IV.

2 The heat equation

We now discuss the problem of heat diffusion by following the same framework as for the wave equation. First, we derive the time-dependent heat equation, and then study the steady-state heat equation in the disc, which leads us back to the basic question (7).

2.1 Derivation of the heat equation

Consider an infinite metal plate which we model as the plane $\mathbb{R}^2$, and suppose we are given an initial heat distribution at time $t = 0$. Let the temperature at the point (x, y) at time t be denoted by $u(x, y, t)$.

Consider a small square centered at (x_0, y_0) with sides parallel to the axis and of side length h, as shown in Figure 9. The amount of heat energy in S at time t is given by

$$H(t) = \sigma \iint_S u(x,y,t)\, dx\, dy\,,$$

where $\sigma > 0$ is a constant called the specific heat of the material. Therefore, the heat flow into S is

$$\frac{\partial H}{\partial t} = \sigma \iint_S \frac{\partial u}{\partial t}\, dx\, dy\,,$$

which is approximately equal to

$$\sigma h^2 \frac{\partial u}{\partial t}(x_0, y_0, t),$$

since the area of S is h^2. Now we apply Newton's law of cooling, which states that heat flows from the higher to lower temperature at a rate proportional to the difference, that is, the gradient.

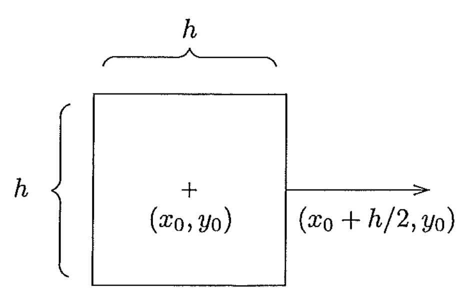

Figure 9. Heat flow through a small square

The heat flow through the vertical side on the right is therefore

$$-\kappa h \frac{\partial u}{\partial x}(x_0 + h/2, y_0, t)\,,$$

where $\kappa > 0$ is the conductivity of the material. A similar argument for the other sides shows that the total heat flow through the square S is given by

$$\kappa h \left[\frac{\partial u}{\partial x}(x_0 + h/2, y_0, t) - \frac{\partial u}{\partial x}(x_0 - h/2, y_0, t) \right.$$
$$\left. + \frac{\partial u}{\partial y}(x_0, y_0 + h/2, t) - \frac{\partial u}{\partial y}(x_0, y_0 - h/2, t) \right].$$

Applying the mean value theorem and letting h tend to zero, we find that

$$\frac{\sigma}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2};$$

this is called the **time-dependent heat equation**, often abbreviated to the heat equation.

2.2 Steady-state heat equation in the disc

After a long period of time, there is no more heat exchange, so that the system reaches thermal equilibrium and $\partial u/\partial t = 0$. In this case, the time-dependent heat equation reduces to the **steady-state heat equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{10}$$

The operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ is of such importance in mathematics and physics that it is often abbreviated as $\triangle$ and given a name: the Laplace operator or **Laplacian**. So the steady-state heat equation is written as

$$\triangle u = 0,$$

and solutions to this equation are called **harmonic functions**.

Consider the unit disc in the plane

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

whose boundary is the unit circle C. In polar coordinates (r, θ), with $0 \leq r$ and $0 \leq \theta < 2\pi$, we have

$$D = \{(r, \theta) : 0 \leq r < 1\} \quad \text{and} \quad C = \{(r, \theta) : r = 1\}.$$

The problem, often called the **Dirichlet problem** (for the Laplacian on the unit disc), is to solve the steady-state heat equation in the unit disc subject to the boundary condition $u = f$ on C. This corresponds to fixing a predetermined temperature distribution on the circle, waiting a long time, and then looking at the temperature distribution inside the disc.

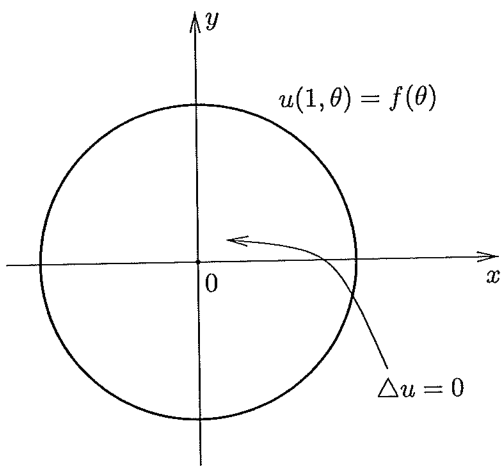

Figure 10. The Dirichlet problem for the disc

While the method of separation of variables will turn out to be useful for equation (10), a difficulty comes from the fact that the boundary condition is not easily expressed in terms of rectangular coordinates. Since this boundary condition is best described by the coordinates (r, θ), namely $u(1, \theta) = f(\theta)$, we rewrite the Laplacian in polar coordinates. An application of the chain rule gives (Exercise 10):

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

We now multiply both sides by r^2, and since $\triangle u = 0$, we get

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}.$$

Separating these variables, and looking for a solution of the form $u(r, \theta) = F(r)G(\theta)$, we find

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}.$$

Since the two sides depend on different variables, they must both be constant, say equal to λ. We therefore get the following equations:

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0\,, \\ r^2 F''(r) + rF'(r) - \lambda F(r) = 0. \end{cases}$$

Since G must be periodic of period 2π, this implies that $\lambda \geq 0$ and (as we have seen before) that $\lambda = m^2$ where m is an integer; hence

$$G(\theta) = \tilde{A} \cos m\theta + \tilde{B} \sin m\theta.$$

An application of Euler's identity, $e^{ix} = \cos x + i \sin x$, allows one to rewrite G in terms of complex exponentials,

$$G(\theta) = Ae^{im\theta} + Be^{-im\theta}.$$

With $\lambda = m^2$ and $m \neq 0$, two simple solutions of the equation in F are $F(r) = r^m$ and $F(r) = r^{-m}$ (Exercise 11 gives further information about these solutions). If $m = 0$, then $F(r) = 1$ and $F(r) = \log r$ are two solutions. If $m > 0$, we note that r^{-m} grows unboundedly large as r tends to zero, so $F(r)G(\theta)$ is unbounded at the origin; the same occurs when $m = 0$ and $F(r) = \log r$. We reject these solutions as contrary to our intuition. Therefore, we are left with the following special functions:

$$u_m(r, \theta) = r^{|m|} e^{im\theta}, \qquad m \in \mathbb{Z}.$$

We now make the important observation that (10) is *linear*, and so as in the case of the vibrating string, we may superpose the above special solutions to obtain the presumed general solution:

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}.$$

If this expression gave all the solutions to the steady-state heat equation, then for a reasonable f we should have

$$u(1, \theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta).$$

We therefore ask again in this context: given any reasonable function f on $[0, 2\pi]$ with $f(0) = f(2\pi)$, can we find coefficients a_m so that

$$f(\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \;\; ?$$

where $\varphi(h) \to 0$ as $h \to 0$.

Deduce that

$$\frac{F(x+h)+F(x-h)-2F(x)}{h^2} \to F''(x) \qquad \text{as } h \to 0.$$

[Hint: This is simply a Taylor expansion. It may be obtained by noting that

$$F(x+h)-F(x) = \int_x^{x+h} F'(y)\,dy,$$

and then writing $F'(y) = F'(x) + (y-x)F''(x) + (y-x)\psi(y-x)$, where $\psi(h) \to 0$ as $h \to 0$.]

9. In the case of the plucked string, use the formula for the Fourier sine coefficients to show that

$$A_m = \frac{2h}{m^2} \frac{\sin mp}{p(\pi - p)}.$$

For what position of p are the second, fourth, ... harmonics missing? For what position of p are the third, sixth, ... harmonics missing?

10. Show that the expression of the Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also, prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

11. Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2F''(r) + rF'(r) - n^2F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

[Hint: If F solves the equation, write $F(r) = g(r)r^n$, find the equation satisfied by g, and conclude that $rg'(r) + 2ng(r) = c$ where c is a constant.]

Corollary 5.3 *If f is integrable on the circle and $\hat{f}(n) = 0$ for all n, then $f = 0$ at all points of continuity of f.*

The proof is immediate since all the partial sums are 0, hence all the Cesàro means are 0.

Corollary 5.4 *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

This means that if f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $\epsilon > 0$, then there exists a trigonometric polynomial P such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } -\pi \leq x \leq \pi.$$

This follows immediately from the theorem since the partial sums, hence the Cesàro means, are trigonometric polynomials. Corollary 5.4 is the periodic analogue of the Weierstrass approximation theorem for polynomials which can be found in Exercise 16.

5.3 Abel means and summation

Another method of summation was first considered by Abel and actually predates the Cesàro method.

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantities $A(r)$ are called the **Abel means** of the series. One can prove that if the series converges to s, then it is Abel summable to s. Moreover, the method of Abel summability is even more powerful than the Cesàro method: when the series is Cesàro summable, it is always Abel summable to the same sum. However, if we consider the series

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1),$$

then one can show that it is Abel summable to $1/4$ since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2},$$

but this series is not Cesàro summable; see Exercise 13.

5.4 The Poisson kernel and Dirichlet's problem in the unit disc

To adapt Abel summability to the context of Fourier series, we define the Abel means of the function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

Since the index n takes positive and negative values, it is natural to write $c_0 = a_0$, and $c_n = a_n e^{in\theta} + a_{-n} e^{-in\theta}$ for $n > 0$, so that the Abel means of the Fourier series correspond to the definition given in the previous section for numerical series.

We note that since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \leq r < 1$. Just as in the case of Cesàro means, the key fact is that these Abel means can be written as convolutions

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ is the **Poisson kernel** given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{4}$$

In fact,

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi}\, d\varphi \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) d\varphi, \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series.

Lemma 5.5 *If* $0 \leq r < 1$, *then*

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

The Poisson kernel is a good kernel,[8] *as* r *tends to* 1 *from below.*

Proof. The identity $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$ has already been derived in Section 1.1. Note that

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta).$$

Hence if $1/2 \leq r \leq 1$ and $\delta \leq |\theta| \leq \pi$, then

$$1 - 2r\cos\theta + r^2 \geq c_\delta > 0.$$

Thus $P_r(\theta) \leq (1-r^2)/c_\delta$ when $\delta \leq |\theta| \leq \pi$, and the third property of good kernels is verified. Clearly $P_r(\theta) \geq 0$, and integrating the expression (4) term by term (which is justified by the absolute convergence of the series) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1,$$

thereby concluding the proof that P_r is a good kernel.

Combining this lemma with Theorem 4.1, we obtain our next result.

Theorem 5.6 *The Fourier series of an integrable function on the circle is Abel summable to* f *at every point of continuity. Moreover, if* f *is continuous on the circle, then the Fourier series of* f *is uniformly Abel summable to* f*.*

We now return to a problem discussed in Chapter 1, where we sketched the solution of the steady-state heat equation $\triangle u = 0$ in the unit disc with boundary condition $u = f$ on the circle. We expressed the Laplacian in terms of polar coordinates, separated variables, and expected that a solution was given by

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}, \tag{5}$$

where a_m was the m^{th} Fourier coefficient of f. In other words, we were led to take

$$u(r,\theta) = A_r(f)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)P_r(\theta-\varphi)\,d\varphi.$$

We are now in a position to show that this is indeed the case.

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[8] In this case, the family of kernels is indexed by a continuous parameter $0 \leq r < 1$, rather than the discrete n considered previously. In the definition of good kernels, we simply replace n by r and take the limit in property (c) appropriately, for example $r \to 1$ in this case.

Theorem 5.7 *Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by the Poisson integral*

$$u(r,\theta) = (f * P_r)(\theta) \tag{6}$$

has the following properties:

(i) *u has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.*

(ii) *If θ is any point of continuity of f, then*

$$\lim_{r \to 1} u(r,\theta) = f(\theta).$$

If f is continuous everywhere, then this limit is uniform.

(iii) *If f is continuous, then $u(r,\theta)$ is the unique solution to the steady-state heat equation in the disc which satisfies conditions* (i) *and* (ii).

Proof. For (i), we recall that the function u is given by the series (5). Fix $\rho < 1$; inside each disc of radius $r < \rho < 1$ centered at the origin, the series for u can be differentiated term by term, and the differentiated series is uniformly and absolutely convergent. Thus u can be differentiated twice (in fact infinitely many times), and since this holds for all $\rho < 1$, we conclude that u is twice differentiable inside the unit disc. Moreover, in polar coordinates,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

so term by term differentiation shows that $\Delta u = 0$.

The proof of (ii) is a simple application of the previous theorem. To prove (iii) we argue as follows. Suppose v solves the steady-state heat equation in the disc and converges to f uniformly as r tends to 1 from below. For each fixed r with $0 < r < 1$, the function $v(r,\theta)$ has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \quad \text{where} \quad a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta) e^{-in\theta}\, d\theta.$$

Taking into account that $v(r,\theta)$ solves the equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \tag{7}$$

we find that

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0. \tag{8}$$

Indeed, we may first multiply (7) by $e^{-in\theta}$ and integrate in θ. Then, since v is periodic, two integrations by parts give

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial\theta^2}(r,\theta)e^{-in\theta}\,d\theta = -n^2 a_n(r).$$

Finally, we may interchange the order of differentiation and integration, which is permissible since v has two continuous derivatives; this yields (8).

Therefore, we must have $a_n(r) = A_n r^n + B_n r^{-n}$ for some constants A_n and B_n, when $n \neq 0$ (see Exercise 11 in Chapter 1). To evaluate the constants, we first observe that each term $a_n(r)$ is bounded because v is bounded, therefore $B_n = 0$. To find A_n we let $r \to 1$. Since v converges uniformly to f as $r \to 1$ we find that

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta.$$

By a similar argument, this formula also holds when $n = 0$. Our conclusion is that for each $0 < r < 1$, the Fourier series of v is given by the series of $u(r,\theta)$, so by the uniqueness of Fourier series for continuous functions, we must have $u = v$.

Remark. By part (iii) of the theorem, we may conclude that if u solves $\triangle u = 0$ in the disc, and converges to 0 uniformly as $r \to 1$, then u must be identically 0. However, if uniform convergence is replaced by pointwise convergence, this conclusion may fail; see Exercise 18.

6 Exercises

1. Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.

18. If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \leq r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\triangle u = 0$ in the disc.

(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ.

However, u is not identically zero.

19. Solve Laplace's equation $\triangle u = 0$ in the semi infinite strip

$$S = \{(x,y) : 0 < x < 1,\ 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0,y) = 0 & \text{when } 0 \leq y, \\ u(1,y) = 0 & \text{when } 0 \leq y, \\ u(x,0) = f(x) & \text{when } 0 \leq x \leq 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y}\sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula (6).

20. Consider the Dirichlet problem in the annulus defined by $\{(r,\theta) : \rho < r < 1\}$, where $0 < \rho < 1$ is the inner radius. The problem is to solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions

$$\begin{cases} u(1,\theta) = f(\theta), \\ u(\rho,\theta) = g(\theta), \end{cases}$$