AN ALMOST-ORTHOGONALITY PRINCIPLE IN L^2 FOR DIRECTIONAL MAXIMAL FUNCTIONS

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ABSTRACT. In this work we improve our result in [2]. We prove a strong-type almost-orthogonality principle for maximal functions along several directions. We use geometric methods and a covering lemma.

1. INTRODUCTION

Let Ω be a subset of $[0,\pi)$. Associated to Ω we consider the basis $\mathcal{B}$ of all rectangles in $\mathbb{R}^2$ whose longest side forms an angle θ with the x-axis, for some $\theta \in \Omega$. The maximal operator associated with the set Ω is defined by

$$M_\Omega f(x) = \sup_{x\in R\in\mathcal{B}} \frac{1}{|R|}\int_R |f(y)|\,dy.$$

The study of directional maximal functions began many years ago, and some particular cases were studied by Strömberg [11], Córdoba and Fefferman [5], Nagel, Stein and Wainger [9], Sjögren and Sjölin [10]. More recently, the interest on these problems was renewed with the results of Barrionuevo [3] and Katz [7, 8]. Nevertheless, only the operators associated to some particular sets Ω are well understood. Namely, the cases of lacunary sets of directions ([9] and [10]) and of finite sets [8].

In [2] we proposed a new method to study this operators. We decomposed Ω into several consecutive blocks, Ω_j. We proved an almost-orthogonality

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2000 Mathematics Subject Classification: 42B25

Key words and phrases: Maximal operators, strong-type estimates.

Research partially supported by EU network 'HARP', and by MCyT Grant BFM2001/0189 .

principle that essentially meant that the weak L^2-norm of M_Ω is the supremum of the norms of the operators M_{Ω_j}, plus a term associated to the sequence of end-points of the blocks. Let us explain this.

Without loss of generality, we can assume that $\Omega \subset [0, \pi/4)$. Let $\Omega_0 = \{\theta_1 > \theta_2 > ... > \theta_j > ...\}$ be an ordered subset of Ω. We take $\theta_0 = \frac{\pi}{4}$ and consider, for each $j \geq 1$, sets $\Omega_j = [\theta_j, \theta_{j-1}) \cap \Omega$, such that $\theta_j \in \Omega_j$ for all j. Assume also that $\Omega = \cup \Omega_j$.

To each one of the sets Ω_j, $j = 0, 1, 2, \ldots$, we associate the corresponding basis $\mathcal{B}_j$, and define the maximal operators associated to the sets Ω_j by

$$M_{\Omega_j} f(x) = \sup_{x \in R \in \mathcal{B}_j} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad j = 0, 1, 2, \ldots$$

In [2] we proved the following result.

Theorem 1. *There exist constants C_1 and C_2, independent of the set Ω, such that*

$$\|M_\Omega\|^2_{L^2 \to L^{2,\infty}} \leq C_1 \sup_{j \geq 1} \|M_{\Omega_j}\|^2_{L^2 \to L^{2,\infty}} + C_2 \, \|M_{\Omega_0}\|^2_{L^2 \to L^{2,\infty}}, \tag{1}$$

where $\|T\|_{L^2 \to L^{2,\infty}}$ denotes the "weak type $(2,2)$" norm of the operator T.

The main result of this paper is the "strong type (2,2)" analogue of Theorem 1.

Theorem 2. *There exists a constant C independent of Ω such that*

$$\|M_\Omega\|_{L^2 \to L^2} \leq \sup_{j \geq 1} \|M_{\Omega_j}\|_{L^2 \to L^2} + C \, \|M_{\Omega_0}\|_{L^2 \to L^2}, \tag{2}$$

where $\|T\|_{L^2 \to L^2}$ denotes the "strong type $(2,2)$" norm of the operator T.

The proof, presented in Section 2, relies on geometric arguments like the ones used in [2], and on a covering lemma by Carbery [4]. A version of this principle for general p, $1 < p \leq \infty$ can be found in [1].

It is worth noting that in Theorem 2, the constant multiplying the supremum of the norms of the M_{Ω_j} is 1. As we shall see, this will allow us to give an alternative proof to the result by Katz [8]. This and other applications of Theorem 2 are presented in Section 3.

2. The proof of Theorem 2.

We first linearize the operators M_Ω and M_{Ω_j}. For any $\alpha \in \mathbb{Z}^2$, Q_α will denote the unit cube centered at α. Given a set $\Lambda \subset [0, \pi/4)$, for each α we choose a rectangle $R_\alpha \in \mathcal{B}_\Lambda$, such that $R_\alpha \supset Q_\alpha$. We define the operator T_Λ as

$$T_\Lambda f(x) = \sum_\alpha \frac{1}{|R_\alpha|} \left(\int_{R_\alpha} f \right) \chi_{Q_\alpha}(x).$$

By definition, one can easily see that

$$T_\Lambda f(x) \leq M_\Lambda f(x), \tag{3}$$

for any choice of rectangles $\{R_\alpha\}$. On the other hand, there is a sequence of linearized operators $\{T_\Lambda f\}$, associated to grids of smaller cubes in $\mathbb{R}^2$, which converge pointwise to $M_\Lambda f$. By scaling invariance, we need only prove (2) with M_Ω replaced by T_Ω.

We shall show this using the following result, proved by Carbery in [4]. For the sake of completeness, we give the proof of this result, at the same time that we check the constants.

Theorem 3. *Let T_Λ be as above. Then T_Λ is of strong type (p,p) if and only if there exist a constant $C_{p'}$, such that for any sequence $\{\lambda_\alpha\} \subset \mathbb{R}_+$, we have*

$$\int \left(\sum_\alpha \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^{p'} \leq C_{p'} \sum_\alpha |\lambda_\alpha|^{p'}. \tag{4}$$

Moreover, the infimum of the constants $(C_{p'})^{1/p'}$ satisfying (4) is $\|T_\Lambda\|_{L^p \to L^p}$.

Proof: If T_Λ is of strong type (p,p), then its adjoint T_Λ^* defined by

$$T_\Lambda^* g(x) = \sum_\alpha \left(\int_{Q_\alpha} g \right) \frac{1}{|R_\alpha|} \chi_{R_\alpha}(x),$$

is of strong type (p', p'), with the same norm. Taking $g = \sum_\alpha \lambda_\alpha \chi_{Q_\alpha}$, we obtain (4) with $C_{p'} = \|T_\Lambda^*\|_{L^{p'}\to L^{p'}}^{p'} = \|T_\Lambda\|_{L^p\to L^p}^{p'}$.

Conversely, if we have (4) then, for all $h \in L^{p'}$, letting $\lambda_\alpha = |\int_{Q_\alpha} h|$, we get

$$\int |T_\Lambda^* h|^{p'} \leq C_{p'} \sum_\alpha \left| \int_{Q_\alpha} h \right|^{p'} \leq C_{p'} \int |h|^{p'}.$$

(Here we have used Jensen's inequality, since $|Q_\alpha| = 1$, and the fact that the Q_α have disjoint interiors). Hence, T_Λ is of strong type (p,p) and its norm is bounded by $(C_{p'})^{1/p'}$.

□

Let us continue with the proof of Theorem 2. We define T_Ω for some choice of rectangles $\{R_\alpha\}$. We only need to prove that inequality (4) is satisfied, with $p = 2$ and $C_2^{1/2} = \sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2} + C\,\|M_{\Omega_0}\|_{L^2\to L^2}$.

Set

$$I^2 = \int \left(\sum_\alpha \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^2 = \int \left(\sum_l \sum_{\alpha:R_\alpha\in\Omega_l} \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^2$$

$$= \int \sum_l \left(\sum_{\alpha:R_\alpha\in\Omega_l} \lambda_\alpha \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^2$$

$$+2 \sum_l \sum_{j<l} \int \sum_{R_\alpha\in\Omega_l} \sum_{R_\beta\in\Omega_j} \lambda_\alpha \lambda_\beta \frac{1}{|R_\alpha||R_\beta|} \chi_{R_\alpha} \chi_{R_\beta}$$

$$= A + B.$$

For the first term we use (3) and Theorem 3 with $p = 2$ and $\Lambda = \Omega_l$. We obtain

$$A \leq \sum_l \|M_{\Omega_l}\|_{L^2\to L^2}^2 \left(\sum_{\alpha:R_\alpha\in\Omega_l} |\lambda_\alpha|^2 \right)$$

$$\leq \left(\sup_l \|M_{\Omega_l}\|_{L^2\to L^2}^2 \right) \left(\sum_l \sum_{\alpha:R_\alpha\in\Omega_l} |\lambda_\alpha|^2 \right)$$

$$\leq \left(\sup_l \|M_{\Omega_l}\|^2_{L^2\to L^2} \right) \left(\sum_\alpha |\lambda_\alpha|^2 \right). \tag{5}$$

Now we have to study B. Using the same geometric arguments as in [2], we have that there exists a constant C such that, if $R_\alpha \in \Omega_l$ and $R_\beta \in \Omega_j$ with $j < l$, then we can find certain rectangles $\widetilde{R}_\alpha^-$ and $\widetilde{R}_\beta^+$, containing R_α and R_β, respectively, pointing in the direction of θ_j and so that

$$\frac{|R_\alpha \cap R_\beta|}{|R_\alpha||R_\beta|} \leq C\frac{|\widetilde{R}_\alpha^- \cap R_\beta|}{|\widetilde{R}_\alpha^-||R_\beta|} + C\frac{|R_\alpha \cap \widetilde{R}_\beta^+|}{|R_\alpha||\widetilde{R}_\beta^+|}.$$

Observe that both $\widetilde{R}_\alpha^-$ and $\widetilde{R}_\beta^+$ are rectangles of the basis $\mathcal{B}_0$. Then,

$$B \leq 2C \sum_l \sum_{j<l} \int \sum_{R_\alpha\in\Omega_l} \sum_{R_\beta\in\Omega_j} \lambda_\alpha\lambda_\beta \frac{1}{|\widetilde{R}_\alpha^-||R_\beta|} \chi_{\widetilde{R}_\alpha^-}\chi_{R_\beta}$$

$$+2C \sum_l \sum_{j<l} \int \sum_{R_\alpha\in\Omega_l} \sum_{R_\beta\in\Omega_j} \lambda_\alpha\lambda_\beta \frac{1}{|R_\alpha||\widetilde{R}_\beta^+|} \chi_{R_\beta}\chi_{\widetilde{R}_\beta^+}$$

$$= B^- + B^+.$$

We shall only work with the B^- (the other term is analogous). So,

$$B = 2C \sum_l \sum_{j<l} \int \sum_{R_\alpha\in\Omega_l} \sum_{R_\beta\in\Omega_j} \lambda_\alpha\lambda_\beta \frac{1}{|\widetilde{R}_\alpha^-||R_\beta|} \chi_{\widetilde{R}_\alpha^-}\chi_{R_\beta}$$

$$\leq 2C \int \left(\sum_l \sum_{R_\alpha\in\Omega_l} \lambda_\alpha \frac{\chi_{\widetilde{R}_\alpha^-}}{|\widetilde{R}_\alpha^-|} \right) \left(\sum_j \sum_{R_\beta\in\Omega_j} \lambda_\beta \frac{\chi_{R_\beta}}{|R_\beta|} \right). \tag{6}$$

We use Cauchy-Schwarz's inequality to bound (6) by

$$\leq 2C \left(\int \left(\sum_l \sum_{R_\alpha\in\Omega_l} \lambda_\alpha \frac{\chi_{\widetilde{R}_\alpha^-}}{|\widetilde{R}_\alpha^-|} \right)^2 \right)^{1/2} \left(\int \left(\sum_j \sum_{R_\beta\in\Omega_j} \lambda_\beta \frac{\chi_{R_\beta}}{|R_\beta|} \right)^2 \right)^{1/2}.$$

Now, notice that $\widetilde{R}_\alpha^- \in \Omega_0$ for all α. Hence,we can majorize the first integral using again Theorem 3 and (3), and obtain

$$B^- \leq 2C\|M_{\Omega_0}\|_{L^2\to L^2} \left(\sum_\alpha |\lambda_\alpha|^2 \right)^{1/2} I, \tag{7}$$

and also the same bound for B^+. Combining the bounds (5) for A and (7) for $B^{\pm}$ we get

$$I^2 \leq \left(\sup_l \|M_{\Omega_l}\|^2_{L^2\to L^2}\right)\left(\sum_\alpha |\lambda_\alpha|^2\right) + C\,\|M_{\Omega_0}\|_{L^2\to L^2}\left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2} I.$$

This implies

$$I \leq \left(\sup_l \|M_{\Omega_l}\|_{L^2\to L^2} + C\|M_{\Omega_0}\|_{L^2\to L^2}\right)\left(\sum_\alpha |\lambda_\alpha|^2\right)^{1/2}.$$

By Theorem 3, this finishes the proof of Theorem 2.

□

3. Some Applications

As a corollary of Theorem 2, we give a simple proof of the following result by Katz [8].

Corollary 4. *There exists a constant K such that, for any set $\Omega \subset [0, \frac{\pi}{4})$ with cardinality $N > 1$, one has*

$$\|M_\Omega\|_{L^2\to L^2} \leq K(\log N). \tag{8}$$

In [2] we obtained the bound $K(\log N)^\alpha$, for some $\alpha > 1$ which depended only on the constants C_1 and C_2 in Theorem 1. Here we are able to obtain the optimal exponent $\alpha = 1$, due to the fact that we have a constant 1 in front of the term $\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2}$ in (2).

Proof: We can assume that $N = 2^M$. We use induction on M. For $M = 1, 2$ the inequality (8) follows from the boundedness of the strong maximal function. Now assume $M \geq 3$ and that (8) is true for all sets with cardinality 2^k where $1 \leq k < M$; we may assume that K is big (indeed we shall need $K \geq 2C/\log 2$ where C is the constant in Theorem 2). If the elements of Ω are ordered, $\{\phi_1 > \phi_2 > \ldots > \phi_N\}$, we define Ω_0 to be the set consisting only on ϕ_N and the middle element $\phi_{\frac{N}{2}}$. In this way, there are only two sets Ω_1 and Ω_2. Each one of them has $N/2$ elements. So by Theorem 2 and the induction hypothesis,

$$\|M_\Omega\|_{L^2\to L^2} \le K \log \frac{N}{2} + 2C = K \log N - K \log 2 + 2C \le K \log N$$

since we had assumed $K \ge \frac{2C}{\log 2}$.

□

In his paper [8], Katz also proves an analogous result to (8) for the weak type of M_Ω. Namely,

$$\|M_\Omega\|_{L^2\to L^{2,\infty}} \le K(\log N)^{1/2}, \tag{9}$$

for any set $\Omega \in [0, \frac{\pi}{4})$ with cardinality N.

In [2], as a corollary of the almost-orthogonality principle (1), we showed that

$$\|M_\Omega\|_{L^2\to L^{2,\infty}} \le K(\log N)^{\beta}, \tag{10}$$

for some $\beta > 1/2$ which depended on C_1 and C_2. If we were able to prove (1) with $C_1 = 1$, the same argument of Corollary 4 would give us the optimal exponent $\beta = 1/2$. With a different argument, Anthony Carbery has shown that an improvement of (10) can be derived from a slight change in the proof of Theorem 2. We include this result here.

We need first the following weak-type analogue of Theorem 3, whose proof can be found in [4].

Theorem 5. *Let T_Λ be as in Theorem 3. Then T_Λ is of weak type $(2,2)$ if and only if there exist a constant C_2, such that for any $A \subset \mathbb{Z}^2$, we have*

$$\int \left(\sum_{\alpha\in A} \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^2 \le C_2(\sharp A). \tag{11}$$

Moreover, if $B_2(T_\Lambda)$ denotes the infimum of all the constants C_2 satisfying (11), then $B_2(T_\Lambda)$ is equivalent to $\|T_\Lambda\|^2_{L^2\to L^{2,\infty}}$.

Corollary 6. *(A. Carbery.) There exists a constant C such that for any set $\Omega \subset [0, \frac{\pi}{4})$ with cardinality $N > 1$, one has*

$$\|M_\Omega\|_{L^2 \to L^{2,\infty}} \le C(\log N)^{1/2}(\log\log N). \tag{12}$$

PROOF: Let us denote by B_N the supremum of $B_2(T_\Lambda)$, the supremum taken on all T_Λ such that the cardinality of Λ is N. Thus, we have to show

$$B_N \le C \log N (\log\log N)^2.$$

We fix Ω of cardinality N and T_Ω. As we did in the proof of Corollary 4, we define Ω_0 as the set consisting only on the last and the middle element in Ω. Consequently, each one of the sets Ω_1 and Ω_2 has $N/2$ elements. Then, a repetition of the proof of Theorem 2 gives

$$\int \left(\sum_{\alpha \in A} \frac{1}{|R_\alpha|} \chi_{R_\alpha} \right)^2 \le B_{N/2}\,(\sharp A)$$

$$+2C \left(\int \left(\sum_l \sum_{R_\alpha \in \Omega_l} \frac{\chi_{\widetilde{R}_\alpha^\pm}}{|\widetilde{R}_\alpha^\pm|} \right)^{p'} \right)^{1/p'} \left(\int \left(\sum_j \sum_{R_\beta \in \Omega_j} \frac{\chi_{R_\beta}}{|R_\beta|} \right)^{p} \right)^{1/p}. \tag{13}$$

Here, instead of applying Cauchy-Schwarz's inequality in (6), we have used Hölder's inequality for some $p < 2$ (which implies $p' > 2$) that will be chosen later. Now, by Theorem 3, the right hand side of (13) is bounded by

$$B_{N/2}\,(\sharp A) + 2C\|M_{\Omega_0}\|_{L^p \to L^p}\,(\sharp A)^{1/p'}\,\|M_\Omega\|_{L^{p'} \to L^{p'}}\,(\sharp A)^{1/p}. \tag{14}$$

By Corollary 4 and interpolation with L^∞,

$$\|M_\Omega\|_{L^{p'} \to L^{p'}} \le C(\log N)^{2/p'} \le \widehat{C}, \tag{15}$$

for some absolute constant $\widehat{C}$, provided that we choose p' such that $\frac{2}{p'} = \frac{1}{\log\log N}$. Thus, (14) and (15) imply that

$$B_N \le B_{N/2} + \widetilde{C}\|M_{\Omega_0}\|_{L^p \to L^p}.$$

Since Ω_0 has only two elements, by Jessen-Marcinkiewicz-Zygmund theorem [6] and our choice of p', we have

$$\|M_{\Omega_0}\|_{L^p \to L^p} \leq \frac{C}{(p-1)^2} \leq C(\log\log N)^2.$$

Applying now an induction argument, we easily obtain that
$B_N \leq C \log N (\log\log N)^2$, and hence (12).

□

ACKNOWLEDGEMENT: We would like to thank Anthony Carbery for many nice and interesting discussions on this subject and for letting us include Corollary 6 in this paper.

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