MATH 857: Homework 1: Due Feb 22

1. Folland, Chapter 6, Ex. 2 and 5.

2. Use induction to prove the following generalization of Hölder's inequality: Let $(X, \mathcal{M}, \mu)$ be a measure space. Given $1 < p_1, p_2, \ldots, p_n < \infty$ such that $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \cdots + \dfrac{1}{p_n} = 1$, and measurable functions $f_j \in L_j^p(X)$, show that

$$\left| \int_X f_1 f_2 \cdots f_n \, d\mu \right| \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}.$$

3. Let $1 \le p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ if and only if $\|f_n\|_p \to \|f\|_p$.

4. If $\mu(X) = 1$, prove that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$, even in the case where $\|f\|_\infty = \infty$. (In fact, the result is also true for any space X with finite measure).

5. Let p be a real number, $0 < p < \infty$.

   (a) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{p<r<\infty} L^r$.

   (b) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{0<r<p} L^r$.

   (c) Find a measure space (X, μ) and a function $f \in L^p \setminus \bigcup_{r\ne p} L^r$.

   Hint: Consider functions of the type $\frac{1}{x^a}$ *or* $\frac{1}{x^a(1+\log x)^b}$ *for appropriate powers* a,b.

6. Let p be a real number, $0 < p < \infty$.

   (a) Find a measure space (X, μ) and a function $f \in \left(\bigcap_{p<r<\infty} L^r \right) \setminus L^p$.

   (b) Find a measure space (X, μ) and a function $f \in \left(\bigcap_{0<r<p} L^r \right) \setminus L^p$.

   (c) Prove that it is impossible to find a measure space (X, μ) and a function $f \in \left(\bigcap_{r\ne p} L^r \right) \setminus L^p$.

7. Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = x(\pi - x)$.

   (a) Draw the graph on f on $[-\pi, \pi]$.

(b) Compute the Fourier coefficients of f and show that

$$f(x) = \frac{8}{\pi} \sum_{k \text{ odd }, k>0} \frac{\sin(kx)}{k^3}.$$

8. Consider the function defined on $[-\pi, \pi]$ by $f(x) = |x|$.

   (a) Draw the graph on f on $[-\pi, \pi]$.

   (b) Compute the Fourier coefficients of f.

   (c) Write the Fourier series of f in terms of sines and cosines.

   (d) Taking $x = 0$, show that

$$\sum_{n \text{ odd }, n>1} \frac{1}{n^2} = \frac{\pi^2}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

9. Consider an interval $[a, b] \subset [-\pi, \pi]$ and $f(x) = \chi_{[a,b]}(x)$.

   (a) Find the Fourier series of f (in terms of exponentials).

   (b) Show that if $a \neq \pi$ or $b \neq \pi$, and $a \neq b$, then the Fourier series does not converge **absolutely** for any x. Hint: Take $x_0 = (b - a)/2$ and check that $|sin(nx_0)| \geq c > 0$ for many values of n.

   (c) However, prove that the Fourier series converges at every point x. What happens in the case $a = -\pi$ and $b = \pi$?