STRONG TYPE INEQUALITIES AND AN ALMOST-ORTHOGONALITY PRINCIPLE FOR FAMILIES OF MAXIMAL OPERATORS ALONG DIRECTIONS IN $\mathbb{R}^2$

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Abstract. In this paper we prove an almost-orthogonality principle for maximal operators over arbitrary sets of directions in $\mathbb{R}^2$. Namely, we obtain L^p-bounds for an operator of this type from the corresponding L^p-bounds of the maximal functions associated to a certain partition of the set of directions, and from the particular structure of this partition. We give applications to several types of maximal operators.

1. Introduction

In this paper we continue the study of maximal functions along directions in $\mathbb{R}^2$, initiated in [1] in connection with a certain notion of 'planar' almost-orthogonality.

Let Ω_0 be an ordered subset of $[0, \frac{\pi}{4})$. We denote its elements by $\theta_1, \theta_2, \ldots$, with

$$\frac{\pi}{4} = \theta_0 > \theta_1 > \theta_2 > ... > \theta_j > ...$$

We shall refer to Ω_0 as the 'separating' set, and to its elements θ_j as the 'separators'. For each $j \geq 1$, we have a set $\Omega_j \subset [\theta_j, \theta_{j-1})$, with $\theta_j \in \Omega_j$. The maximal operators associated to these sets are defined as

$$M_{\Omega_0} f(x,y) = \sup_{h>0, j\geq 1} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\theta_j, y - t\sin\theta_j)\, dt \right|,$$

and

$$M_{\Omega_j} f(x,y) = \sup_{h>0, \theta\in\Omega_j} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\theta, y - t\sin\theta)\, dt \right|,$$

for $j \geq 1$. The maximal function over all these directions, that is, on $\Omega = \cup_{j\geq 1}\Omega_j$ is

$$M_\Omega f(x,y) = \sup_{j\geq 1} M_{\Omega_j} f(x,y).$$

We want to determine L^p-bounds of M_Ω from the corresponding L^p-bounds of each M_{Ω_j} and the particular structure of the separating set Ω_0. In [1], the author proved with F. Soria and A. Vargas the following relationship

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between the L^2-norms of M_Ω and M_{Ω_j}: there exist constants C_1 and C_2, independent of the set Ω, such that

$$(1)\qquad \|M_\Omega\|^2_{L^2\to L^{2,\infty}} \le C_1 \sup_{j\ge 1} \|M_{\Omega_j}\|^2_{L^2\to L^{2,\infty}} + C_2\, \|M_{\Omega_0}\|^2_{L^2\to L^{2,\infty}}.$$

As a corollary, one gets a simple proof of a beautiful result by N. Katz [7], solving a conjecture that was open for many years: if Ω has cardinality $N>1$, then

$$\|M_\Omega\|_{L^2\to L^{2,\infty}} \le C(\log N)^\alpha,$$

for some constants C and α independent of Ω. Actually, Katz proves it with the sharp exponent $\alpha = \frac{1}{2}$.

Another direct consequence of (1) is the following. If Ω_0 is a lacunary sequence, then

$$(2)\qquad \|\sup M_{\Omega_j}\|_{L^2\to L^{2,\infty}} \le C \sup \|M_{\Omega_j}\|_{L^2\to L^{2,\infty}},$$

i.e. we bound the norm of a supremum by the supremum of the norms. Notice that we cannot directly get strong type 2 inequalities by interpolation between this and estimates on L^1, since $\cup_{j\ge 1}\Omega_j$ is an infinite set.

Inequalities (1) and (2) show that the operators M_{Ω_j} satisfy an almost-orthogonality principle when we look at their weak type L^2 norms. Our aim is to find a similar relation between the strong-type norms of these maximal functions. Instead of the geometrical arguments used in [1], we shall follow the approach of Nagel, Stein, Wainger [8] (see also [9]). In order to do that, we need to introduce a Littlewood-Paley decomposition and the associated square function. All this work is presented in Section 2. The last section contains several applications.

2. The Main Result

Let $\Omega_0 = \{\theta_1 > \theta_2 > ... > \theta_j > ...\}$ be a subset of $[0, \frac{\pi}{4})$. For each $j \ge 1$, we consider sets $\Omega_j \subset [\theta_j, \theta_{j-1})$, where we take $\theta_0 = \frac{\pi}{4}$. We shall also assume that $\theta_j \in \Omega_j$. We define the maximal operators associated to the sets Ω_j, $j \ge 0$, as

$$M_{\Omega_0} f(x,y) = \sup_{h>0,j} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\theta_j, y - t\sin\theta_j)\, dt \right|,$$

and, if $j \ge 1$

$$M_{\Omega_j} f(x,y) = \sup_{h>0,\theta\in\Omega_j} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\theta, y - t\sin\theta)\, dt \right|,$$

for $f \in S$. We also define the maximal function

$$M_\Omega f(x,y) = \sup_{j\ge 1} M_{\Omega_j} f(x,y).$$

In order to state our result, we need to introduce a certain square function associated with Ω_0. For each $j \ge 1$, set $\delta_j = |\theta_{j-1} - \theta_j|$. Let us consider the following angular sectors

$$\Delta_j = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \theta_j - \frac{1}{20}\delta_j \le \arctan\left(\frac{x}{-y}\right) < \theta_{j-1} + \frac{1}{20}\delta_j \right\},$$

and the wider sectors

$$\widetilde{\Delta_j} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \theta_j - \frac{1}{10}\delta_j \le \arctan\left(\frac{x}{-y}\right) < \theta_{j-1} + \frac{1}{10}\delta_j \right\}.$$

Given $j \ge 1$, we pick a function ω_j, homogeneous of degree zero, C^∞ on S^1, identically equal to 1 in Δ_j and vanishing outside $\widetilde{\Delta_j}$. The multiplier operator S_j associated to ω_j is defined by

$$(S_j f)\widehat{\ } = \omega_j \widehat{f},$$

and the square function S by

$$Sf(x) = \left(\sum_{j \ge 1} |S_j f(x)|^2 \right)^{1/2}.$$

The properties of S depend in a direct way on the geometry of Ω_0.

We can now state the following two results.

Theorem 1. *With the above notation, given $2 \le p < \infty$, there exists a finite constant C_p such that*

$$\|M_\Omega f\|_p \le C_p \left[\|M_{\Omega_0} f\|_p + \left(\sup_{j \ge 1} \|M_{\Omega_j}\|_{L^p \to L^p} \right) \|Sf\|_p \right]. \tag{3}$$

For $1 < p < 2$, we get the following

Theorem 2. *If $1 < p < 2$, there exists a finite constant C_p such that*

$$\|M_\Omega f\|_p \le C_p \left[\|M_{\Omega_0}\|_{L^p \to L^p} + \left(\sup_{j \ge 1} \|M_{\Omega_j}\|_{L^p \to L^p} \right) \|S\|_{L^p \to L^p}^{2/p} \right] \|f\|_p. \tag{4}$$

Theorem 1 will follow from the two lemmas below. Let us define the directional Hardy-Littlewood maximal function, in the direction of θ by

$$M_\theta f(x) = \sup_{h>0} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\theta, y - t\sin\theta)\,dt \right|.$$

Then we have

Lemma 3. *For each $j \ge 1$ and for all $\theta \in \Omega_j$,*

$$M_\theta f(x) \le C[M_{\theta_j} f(x) + M M_\theta (S_j f)(x)], \tag{5}$$

where M is the ordinary Hardy-Littlewood maximal function.

Proof. Our argument represents a slight modification of the proof of Lemma 3 in [5]. Choose a positive function $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ on $[-1,1]$. Fix $j \geq 1$ and $\theta \in \Omega_j$.

Then if $f \geq 0$,

$$M_\theta f(x) \leq C \sup_{h>0} \frac{1}{2h} \int_{-\infty}^{\infty} \psi\left(\frac{t}{h}\right) f(x - t\cos\theta, y - t\sin\theta)\, dt =$$

$$= C \sup_{h>0} N_{h,j,\theta} f(x). \tag{6}$$

Set $m = \widehat{\psi}$. Take a function $\phi \in C_0^\infty(\mathbb{R}^2)$, with $\phi(\xi) = 1$ if $|\xi| \leq 1$. We decompose $(N_{h,j,\theta} f)\widehat{\ }(\xi)$ as

$$\begin{aligned}(N_{h,j,\theta} f)\widehat{\ }(\xi) &= m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)\widehat{f}(\xi) = \\ &= m(h\xi_1\cos\theta + h\xi_2 \sin\theta)\, \phi(h\delta_j\, \xi)\, \widehat{f}(\xi) + \\ &+ m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)\,(1 - \phi(h\delta_j\, \xi))\,(1 - \omega_j(h\xi))\widehat{f}(\xi) + \\ &+ m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)\,(1 - \phi(h\delta_j\, \xi))\, \omega_j(h\xi)\widehat{f}(\xi) = \\ &= \widehat{I_{h,j,\theta}(f)}(\xi) + \widehat{II_{h,j,\theta}(f)}(\xi) + \widehat{III_{h,j,\theta}(f)}(\xi).\end{aligned} \tag{7}$$

Consider the multiplier in the first term, $m(h\xi_1 \cos\theta + h\xi_2 \sin\theta)\phi(h\delta_j \xi)$. If we compose it with an appropriate rotation, we can write it in the form $m(h\eta_1)\phi(h\delta_j \eta)$. Then, differentiating with respect to η_1 and η_2, we see that the Fourier transform K_1 of $m(h\eta_1)\phi(h\delta_j\, \eta)$ satisfies

$$|z_1|^\alpha |z_2|^\beta |K_1(z)| \leq C\delta_j^{-1+\beta},$$

so that the operator $K_1 * f$ is bounded by the maximal function over rectangles of eccentricity δ_j and sides parallel to the coordinate axes. Therefore $\sup_h I_{h,j,\theta}(f)$ is bounded by $M_{\theta,\delta_j} f$, the maximal function over rectangles of eccentricity δ_j and sides parallel to the direction θ. Now, observe that $|\theta - \theta_j| \leq \delta_j$, so the rectangles in the definition of $M_{\theta,\delta_j} f$ can be included in rectangles of comparable area and sides parallel to θ_j. Hence,

$$|\sup_{h>0} I_{h,j,\theta} f(x)| \leq C M_{\theta_j} f(x).$$

The second term is treated in the same way. The third term is clearly controlled by $MM_\theta(S_j f)$. This finishes the proof of Lemma 3.

□

For the proof of Theorem 1, we also need the following lemma.

Lemma 4. *Assume that for some $p > 1$, $q \geq 2$ and for any sequence of functions $\{f_j\}$, one has*

$$\left\| \left(\sum_{j=1}^{\infty} \left| M_{\Omega_j} f_j \right|^q \right)^{1/q} \right\|_p \leq B \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_p . \tag{8}$$

Then

$$\|M_\Omega f\|_p \le C_p\,[\,\|M_{\Omega_0} f\|_p + B\|Sf\|_p\,]\,, \tag{9}$$

for some constant C_p.

Proof. From the pointwise estimate (5) in Lemma 3, we get

$$M_\Omega f(x) = \sup_{j\ge1,\theta\in\Omega_j} M_\theta f(x) \le C\left[M_{\Omega_0} f(x) + \sup_{j\ge1,\theta\in\Omega_j} MM_\theta(S_j f)(x)\right].$$

Using the Hardy-Littlewood maximal theorem, we have

$$\left\|\sup_{j\ge1,\theta\in\Omega_j} MM_\theta(S_j f)\right\|_p \le C_p\left\|\sup_{j\ge1,\theta\in\Omega_j} M_\theta(S_j f)\right\|_p.$$

Notice that

$$\sup_{j\ge1,\theta\in\Omega_j} M_\theta(S_j f) \le \left(\sum_{j=1}^{\infty}\sup_{\theta\in\Omega_j}(M_\theta(S_j f))^q\right)^{1/q} =$$

$$= \left(\sum_{j=1}^{\infty}\left(M_{\Omega_j}(S_j f)\right)^q\right)^{1/q}.$$

The hypothesis of the lemma implies now

$$\left\|\sup_{j\ge1,\theta\in\Omega_j} M_\theta(S_j f)\right\|_p \le B\left\|\left(\sum_{j=1}^{\infty}|S_j f|^q\right)^{1/q}\right\|_p \le$$

$$\le B\left\|\left(\sum_{j=1}^{\infty}|S_j f|^2\right)^{1/2}\right\|_p = B\,\|Sf\|_p\,.$$

In the second inequality we have used that $q \ge 2$. This gives (9) and, therefore, Lemma 4.

□

Proof of Theorem 1. It is a simple consequence of Lemma 4, from the trivial observation that (8) holds with $B = \sup\|M_{\Omega_j}\|_{L^p\to L^p}$ and $p = q$.

□

Proof of Theorem 2. For technical reasons, we shall prove Theorem 2 for a slightly different square function, $\widetilde{S}$, adapted to the sectors $\widetilde{\Delta_j}$. We take a function $\widetilde{\omega}_j$ identically equal to 1 in $\widetilde{\Delta_j}$ and vanishing outside a slightly wider sector. Then we define the operator $\widetilde{S_j}$ by

$$(\widetilde{S_j}f)\widehat{\ } = \widetilde{\omega_j}\widehat{f},$$

and consider the associated square function $\widetilde{S}$.

The proof follows the main circle of ideas that one can find in [3]. We start by considering the operator $N_{h,j,\theta}f(x)$ defined in the proof of Lemma 3 by (6). As we did there, we split it into three terms, by (7).

We already know that

$$\sup_{h,j\geq 1,\theta\in\Omega_j}(I_{h,j,\theta}(f)+II_{h,j,\theta}(f))\leq M_{\Omega_0}f. \tag{10}$$

Without loss of generality, we can assume that Ω is a finite set. Then there exists a minimal constant $C(\Omega)$ such that

$$\left\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(f)|\right\|_p\leq C(\Omega)\,\|f\|_p.$$

Let us take a sequence of functions $\{g_j\}$. By (7) and (10),

$$\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|\leq$$

$$\sup_{h,j\geq 1,\theta\in\Omega_j}|N_{h,j,\theta}(g_j)|+\sup_{h,j\geq 1,\theta\in\Omega_j}|I_{h,j,\theta}(g_j)+II_{h,j,\theta}(g_j)|\leq$$

$$\leq\sup_{h,j\geq 1,\theta\in\Omega_j}\left|N_{h,j,\theta}(\sup_j|g_j|)\right|+M_{\Omega_0}\left(\sup_{j\geq 1}|g_j|\right)\leq$$

$$\leq\sup_{h,j\geq 1,\theta\in\Omega_j}\left|III_{h,j,\theta}(\sup_{j\geq 1}|g_j|)\right|+2\,M_{\Omega_0}\left(\sup_{j\geq 1}|g_j|\right).$$

Therefore,

$$\left\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|\right\|_p\leq(C(\Omega)+2\,\|M_{\Omega_0}\|_{L^p\to L^p})\left\|\sup_{j\geq 1}|g_j|\right\|_p.$$

On the other hand, we have

$$\left\|\left(\sum_{j\geq 1}\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}(g_j)|^p\right)^{1/p}\right\|_p\leq$$

$$\left(\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\right)\left\|\left(\sum_{j\geq 1}|g_j|^p\right)^{1/p}\right\|_p.$$

By interpolation (taking $\theta=\frac{p}{2}$),

$$\left\|\left(\sum_{j\geq 1}\sup_h|III_{h,j,\theta}(g_j)|^2\right)^{1/2}\right\|_p\leq \tag{11}$$

$$(C(\Omega)+2\ \|M_{\Omega_0}\|_{L^p\to L^p})^{1-\theta}\left(\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\right)^{\theta}\left\|\left(\sum_{j\geq 1}|g_j|^2\right)^{1/2}\right\|_p .$$

Now,

$$\left\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(f)|\right\|_p \leq$$

$$\left\|\sup_{h,j\geq 1,\theta\in\Omega_j}|III_{h,j,\theta}(I-\widetilde{S_j})(f)|\right\|_p+\left\|\left(\sum_{j\geq 1}\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}(\widetilde{S_j}f)|^2\right)^{1/2}\right\|_p .$$

The first term in the last expression is 0, because of the definition of $III_{h,j,\theta}$ and $\widetilde{S_j}$. By (11), the second term is bounded by

$$(C(\Omega)+2\ \|M_{\Omega_0}\|_{L^p\to L^p})^{1-\theta}\left(\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\right)^{\theta}\left\|\widetilde{S}f\right\|_p .$$

By the minimality of $C(\Omega)$,

$$C(\Omega)\leq$$

(12)

$$(C(\Omega)+2\|M_{\Omega_0}\|_{L^p\to L^p})^{1-\theta}\left(\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\right)^{\theta}\left\|\widetilde{S}\right\|_{L^p\to L^p} .$$

From equation (12), we derive the following expression for $C(\Omega)$

$$C(\Omega)\leq 2\left[\|M_{\Omega_0}\|_{L^p\to L^p}+\left(\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\right)\ \|\widetilde{S}\|^{2/p}_{L^p\to L^p}\right] .$$

Now, we notice that

$$III_{h,j,\theta}f\leq M(N_{h,j,\theta}(\widetilde{\omega_j}^{\vee}*f)),$$

where M is the Hardy-Littlewood maximal function. The Hardy-Littlewood maximal theorem and the uniform boundedness of the L^1-norm of the $\widetilde{\omega_j}^{\vee}$ imply that

$$\sup_{j\geq 1}\left\|\sup_{h,\theta\in\Omega_j}|III_{h,j,\theta}|\right\|_{L^p\to L^p}\leq C_p\sup_{j\geq 1}\|M_{\Omega_j}\|_{L^p\to L^p},$$

and thus Theorem 2 is proved.

□

3. Some applications

Our first application of Theorem 1 will be a generalization of the result by Nagel, Stein and Wainger [8]. Let $\Omega_0 = \{\theta_j\}_{j\geq 1}$ be a lacunary sequence of numbers in $[0, \frac{\pi}{4})$ tending to zero, that is, assume that there exists $0 < \lambda < 1$ such that $0 \leq \theta_{j+1} \leq \lambda\theta_j$ for all $j \in \mathbb{N}$. The above authors proved that M_{Ω_0} is bounded in L^p for all $1 < p \leq \infty$. We shall show that we can introduce new directions Ω_j between the lacunary separators of Ω_0 and still get a bounded operator.

Theorem 5. *Let Ω_0 be a lacunary sequence. Then, under the hypothesis of Theorem 1,*

$$\|M_\Omega f\|_p \leq C_p \left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^p\to L^p} \right) \|f\|_p \,, \quad 1 < p < \infty. \tag{13}$$

Theorem 5 also generalizes the result by Sjögren and Sjölin [9]. In that paper they proved the same estimate but in the special case in which Ω_j is given by a lacunary sequence, for all j. Our result does not depend on the particular structure of the Ω_j's, and it shows, in a neat way, that the orthogonality property mentioned in the introduction holds when the separating set is lacunary.

Proof. Without loss of generality, we can assume that $\frac{1}{2} < \lambda < 1$. We shall also assume that there exists $0 < \lambda_0 < \lambda$ such that $\theta_{j+1} \geq \lambda_0\theta_j$ for all j. Actually we can take $\lambda_0 = \lambda^2$, and add some terms to the initial sequence in order to get this. Obviously, the maximal operator associated to the new sequence is greater than the maximal operator of the former one. We need this lower bound λ_0 in order to ensure that, for every $j \geq 1$, the sectors $\widetilde{\Delta_j}$ and $\widetilde{\Delta_{j+2}}$ do not overlap.

To get (13), we apply Theorems 1 and 2. In our case, because of Nagel, Stein and Wainger's result, $\|M_{\Omega_0}\|_{L^p\to L^p} \leq C_p$, and so we get

$$\|M_\Omega f\|_p \leq C_p \|f\|_p + C_p \left(\sup_j \|M_{\Omega_j}\|_{L^p\to L^p} \right) \|Sf\|_p \,, \quad \text{if } 2 \leq p < \infty,$$

and

$$\|M_\Omega f\|_p \leq C_p \|f\|_p + C_p \left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^p\to L^p} \right) \|S\|^{2/p}_{L^p\to L^p} \|f\|_p \,, \quad \text{if } 1 < p < 2.$$

Finally, a standard argument, as in [8], gives

$$\|Sf\|_p \leq C_p \|f\|_p \,, \; 1 < p < \infty,$$

and this proves (13).

□

Theorem 5, in turn, gives us the following application: let $\{\theta_j\}_{j\geq 1}$ be a lacunary sequence as above and take, for each j, a family of N directions $\{\alpha_{j,k}\}_{k=1,\dots,N} \subset [\theta_j, \theta_{j-1})$. Define, as usual, the maximal operators M_{Ω_j} by

$$M_{\Omega_j} f(x,y) = \sup_{h>0} \sup_{k=1,\dots,N} \left| \frac{1}{2h} \int_{-h}^{h} f(x - t\cos\alpha_{j,k}, y - t\sin\alpha_{j,k})\, dt \right|,$$

and

$$M_{\Omega} f(x,y) = \sup_{j\geq 1} M_{\Omega_j} f(x,y).$$

Using Nets Katz's result [7], we have

$$\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2} \leq C \log N.$$

We also have the trivial bound

$$\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^1\to L^{1,\infty}} \leq CN.$$

This give us, by interpolation,

$$\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^p\to L^p} \leq C_p N^{\frac{2}{p}-1} (\log N)^{2-\frac{2}{p}}, \quad 1 < p \leq 2.$$

Then, using (13), we get

$$\|M_{\Omega} f\|_p \leq C_p\, N^{\frac{2}{p}-1} (\log N)^{2-\frac{2}{p}}\ \|f\|_p\,, \quad 1 < p \leq 2.$$

Let us now see some other consequences of Theorem 1.

Theorem 6. *Let Ω_0 be a set of N uniformly distributed directions, and assume that $\left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2}\right) < \infty$. Then*

$$\|M_{\Omega} f\|_2 \leq \left[C_1 \log N + C_2 \left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2} \right) \right] \|f\|_2,$$

for some universal constants C_1, C_2.

Proof. It is just a consequence of Theorem 1. Notice that

$$\|M_{\Omega_0}\|_{L^2\to L^2} \leq C \log N,$$

as Strömberg proved in [11]. We also use the fact that, in this case, $\|Sf\|_2 \sim \|f\|_2$ by Plancherel.

□

Remark. A similar result can be obtained for other values of $p \neq 2$. In particular, for $2 \leq p \leq 4$, we can use Cordoba's bound for the square function associated to these directions (see [4]). We have restricted ourselves to the case $p = 2$ for the sake of simplicity.

We shall now consider more general sets Ω_0. In order to do this, we need some definitions. Given an open set $U \subset \mathbb{R}$, we say that a set of intervals $\{I_\beta\}$ is a Whitney decomposition for U (with constant C) if

$$\begin{array}{lll} (i) & I_\beta \subset U, & \text{for all } \beta. \\ (ii) & \bigcup I_\beta = U. & \\ (iii) & \frac{1}{C}|I_\beta| \leq d(I_\beta, \partial U) \leq C|I_\beta|. & \end{array}$$

We shall say that a set S of finite cardinality N is of Whitney type with constants (C_1, C_2) if for each $s \in S$ there is a Whitney decomposition $\{I^s_\beta\}$ of $\{s\}^c$ (with constant C_1) such that at most $C_2(\log N)^2$ of the I^s_β have nonempty intersection with S. This type of set was introduced by Katz in [6].

Theorem 7. *Assume Ω_0 is a set of Whitney type of cardinality $N > 1$. Then*

$$\|M_\Omega f\|_2 \leq C \log N \left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2} \right) \|f\|_2.$$

Proof. Since Ω_0 has cardinality N, $\|M_{\Omega_0}\|_{L^2\to L^2} \leq C \log N$, by Katz's result [7]. In order to get the conclusion from Theorem 1, we must find an appropriate bound for the square function term in (3). We shall show that, for all ξ,

$$\sum_j \omega_j(\xi) \leq C(\log N)^2.$$

This estimate, together with Plancherel's theorem, will prove Theorem 7.

Given $\theta_k \in \Omega_0 \cup \theta_0$, let $\xi_k = (\cos\theta_k, \sin\theta_k)$. Note that if $\theta_{k-1} > \theta \geq \theta_k$ and $\xi = (\cos\theta, \sin\theta)$, then

$$\sum_j \omega_j(\xi) \leq \sum_j \omega_j(\xi_{k-1}) + \sum_j \omega_j(\xi_k),$$

so we need only prove $\sum_j \omega_j(\xi_k) \leq C(\log N)^2$ for all $k = 1, \ldots, N$.

Fix k and consider the Whitney decomposition $\{I^k_\beta\}$ of $\{\theta_k\}^c$ given by the definition of a set of Whitney type. We shall show that for any fixed β,

$$\sum_{j:\theta_j \in I^k_\beta} \omega_j(\xi_k) \leq C, \tag{14}$$

and this will imply that

$$\sum_j \omega_j(\xi_k) = 1 + \sum_\beta \sum_{j:\theta_j\in I_\beta} \omega_j(\xi_k) = 1 + C(\log N)^2.$$

Let us prove (14). Observe that the width of the support of ω_j is less than $\frac{3}{2}(\theta_{j-1} - \theta_j)$. So if

$$(\theta_{j-1} - \theta_j) \leq \frac{2}{3} \min(|\theta_k - \theta_j|, |\theta_k - \theta_{j-1}|),$$

then $\omega_j(\theta_k) = 0$. On the other hand, if the reverse inequality is true, then

$$(\theta_{j-1} - \theta_j) > \frac{2}{3}\min(|\theta_k - \theta_j|, |\theta_k - \theta_{j-1}| \geq \frac{2}{3C_1}|I_\beta^k|.$$

Now, since the intervals $[\theta_j, \theta_{j-1})$ are pairwise disjoint, there are at most $[\frac{3C_1}{2}] + 1$ of those $\theta_j \in I_\beta^k$. Hence (14) is proved. This finishes the proof of Theorem 7.

□

Finally, let us see a result in the spirit of the work of Barrionuevo [2].

Theorem 8. *There is a constant $C_0 > 0$ such that for every Ω_0 with $\sharp(\Omega_0) = N$, we have*

$$\|M_\Omega\|_{L^2\to L^2} \leq C_0 \left(\sup_{j\geq 1} \|M_{\Omega_j}\|_{L^2\to L^2}\right) N^{\frac{C_0}{\sqrt{\log N}}}. \tag{15}$$

Proof. The proof will be by induction on N. If C_0 is big enough, (15) is true for small N. Now, given a small ϵ to be determined later, we shall say $\theta_l \in \Omega_0$ is 'bad' if

$$\sum_j \omega_j(\theta_l) \geq N^\epsilon,$$

i.e. the overlapping of the ω_j's on θ_l is high.

For a fixed bad θ_l, we consider the indices $j(1) < j(2) < \ldots < j(k) < \ldots < l$, such that $\omega_j(\theta_l) \neq 0$. This means that

$$\theta_l \in \left[\theta_{j(k)} - \frac{\delta_{j(k)}}{20}, \theta_{j(k)-1} + \frac{\delta_{j(k)}}{20}\right].$$

Then $\theta_{j(k)}$ form a lacunary sequence tending to θ_l. Indeed,

$$|\theta_l - \theta_{j(k)}| < \frac{1}{20}|\theta_{j(k)} - \theta_{j(k)-1}| < \frac{1}{20}|\theta_{j(k)} - \theta_{j(k-1)}|,$$

and, as $j(k) < l$ for all k, this implies

$$|\theta_{j(k)} - \theta_{j(k-1)}| \leq |\theta_l - \theta_{j(k-1)}| \leq \frac{1}{20}|\theta_{j(k-1)} - \theta_{j(k-2)}|.$$

We define the sets

$$G_-(\theta_l) = \{j < l : \omega_j(\theta_l) \neq 0\},$$
$$G_+(\theta_l) = \{j > l : \omega_j(\theta_l) \neq 0\}.$$

Then, for a 'bad' θ_l, $\sharp(G_-(\theta_l) \cup G_+(\theta_l)) \geq N^\epsilon$. Set $\eta_1^- = \theta_{l(1)}$, where

$$l(1) = \min\{k : \sharp(G_-(\theta_k)) \geq N^\epsilon\}.$$

Next set $\eta_2^- = \theta_{l(2)}$, where

$$l(2) = \min\{k : \theta_k \notin G_-(\eta_1^-) \text{ and } \sharp[G_-(\theta_k) \setminus G(\eta_1^-)] \geq N^\epsilon\}.$$

and we proceed by induction. For each k, the set $G_-(\eta_k^-)$ is a lacunary sequence. We similarly define η_k^+, associated to the sets G_+. The set

$\{\eta_k^+, \eta_k^- : k \geq 1\}$ has at most $2N^{1-\epsilon}$ elements. Then, applying Theorem 5, we have that for each k,

$$\|\sup_{j\in G_\pm(\eta_k^\pm)} M_{\Omega_j}\|_{L^2\to L^2} \leq C' \sup_{j\in G_\pm(\eta_k^\pm)} \|M_{\Omega_j}\|_{L^2\to L^2}. \tag{16}$$

Now we use the induction hypothesis on the cardinality of the separating set, taking instead of Ω_0 the set $\{\eta_k^\pm\}_k$ which, as we pointed out, has at most $2N^{1-\epsilon}$ elements. Thus,

$$\|\sup_k \sup_{j\in G_\pm(\eta_k^\pm)} M_{\Omega_j}\|_{L^2\to L^2} \leq C_0 C' \sup \|M_{\Omega_j}\|_{L^2\to L^2}(N^{1-\epsilon})^\epsilon. \tag{17}$$

Set $G = \big(\cup_k G_-(\eta_k^-)\big) \cup \big(\cup_k G_+(\eta_k^+)\big)$. Then $\sum_{j\notin G} \omega_j(\theta) < N^\epsilon$ for all θ. Hence Lemma 3 and an argument as in the proof of Theorem 7 give

$$\|\sup_{l\notin G} M_{\Omega_l}\|_{L^2\to L^2} \leq C'' N^\epsilon \sup \|M_{\Omega_l}\|_{L^2\to L^2}. \tag{18}$$

Combining (17) and (18), we get

$$\|\sup_j M_{\Omega_j}\|_{L^2\to L^2} \leq \big(C_0 C'(N^{1-\epsilon})^\epsilon + C'' N^\epsilon\big) \sup \|M_{\Omega_j}\|_{L^2\to L^2}.$$

If we now choose $\epsilon = \frac{C_0}{\sqrt{\log N}}$, we have $C_0 C' N^{-\epsilon^2} + C'' \leq C_0$, for big C_0 depending on C', C'', and (15) is proved.

□

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