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Analysis and geometry near the unit ball: proofs, counterexamples, and open questions

Abstract: We present several theorems, counterexamples, and open questions related to convex bodies close to the unit ball. The techniques include spherical harmonic decomposition and some elements of perturbation theory. We hope that this short survey will attract the attention of both young and mature researchers who will be able to surpass our results and resolve some questions we left unanswered.

Keywords: Star bodies, convex bodies, sections, projections, spherical harmonics, spherical Radon transform

MSC 2020: 52A20, 52A38, 44A12, 33C55

1 Introduction

Among convex bodies, the Euclidean ball B is distinguished by several remarkable properties. The most obvious one is that it is perfectly round in the sense that any characteristic of a convex body that, generally speaking, depends on the direction (width, central cross-section area, projection area, etc.) stays constant for B. This property alone has been a source of numerous questions (some resolved and some still open) of the type "If a convex body is round in some particular sense, is it necessarily a ball?"

The second, slightly less obvious, property is that the ball is an extremizer in various minimization and maximization problems in convex geometry, the most famous of which is, probably, the isoperimetric inequality.

Finally, the unit sphere is essentially the only example of the boundary of a convex body on which harmonic analysis is not only possible in principle, but also rich and well developed. This allows one to use various tools from harmonic and functional anal-

Acknowledgement: The first author is supported in part by the Simons Foundation Grant 711907. The second and third authors are supported in part by U. S. National Science Foundation Grants DMS-1900008 and DMS-1600753. The fourth author is supported in part by NSERC.

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https://doi.org/10.1515/9783110775389-011

ysis when dealing with problems whose formulations have nothing to do with spherical harmonics, Hilbert spaces, or operator eigenvalues.

In this chapter, we will endeavor to show the reader a few tricks and techniques from perturbation theory near the unit ball. We have chosen the perturbative regime as both the easiest one to explain and the only one for which we have gained some decent understanding. In a certain sense it is quite natural: if one has some property satisfied by B or some inequality for which B is presumed to be an extremizer, it is quite tempting to ask if the same property can be preserved by a small perturbation of B or if B is at least a local extremizer. It often turns out that the local version of the question is much easier than the global one and can be answered completely. In our opinion, such investigation must be carried out every time a new conjecture is set forth, though we could not find any trace of it in the literature known to us. This curious fact served as a motivation for several recent projects of ours (often together with other people), some of which we attempt to summarize in this survey.

2 Notation and preliminary observations

Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. The radial function $\varrho = \varrho_K : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$\varrho(e) = \max\{t > 0 : te \in K\}.$$

The support function $\mathfrak{h} = \mathfrak{h}_K : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$\mathfrak{h}(e) = \max\{\langle x, e\rangle : x \in K\}$$

(note that the same formula makes sense for all $e \in \mathbb{R}^n$ and gives a 1-homogeneous extension of $\mathfrak{h}$ to the entire space). Their geometric meanings are the length of the longest interval starting at the origin in the direction e that is contained in K and the distance from the origin to the support hyperplane of K parallel to

$$e^{\perp} = \{y \in \mathbb{R}^n : \langle y, e\rangle = 0\}$$

in the direction e.

We always have $\varrho_K \le \mathfrak{h}_K$. For the unit ball centered at the origin, we have $\varrho = \mathfrak{h} \equiv 1$.

The closeness of a convex body K to the unit ball will be usually measured in the Hausdorff distance

$$d(K, L) = \inf\{r > 0 : K + rB \supset L,\ L + rB \supset K\}.$$

When $L = B$, the inequality $d(K, B) \le \varepsilon < 1$ merely means that

$$(1 - \varepsilon)B \subset K \subset (1 + \varepsilon)B.$$

In the case where the formulation of the problem is invariant under linear transformations, a more natural distance to consider is the Banach–Mazur one: $d_{BM}(K, L) = \log\inf\{R > 1 : L \subset TK \subset \text{ RL for some linear transformation } T \text{ of } \mathbb{R}^n\}$. However, if $d_{BM}(K, B) < \varepsilon$, then, replacing K by an appropriate linear image TK, we get $B \subset TK \subset e^{\varepsilon}B$, so $d(TK, B) \le e^{\varepsilon} - 1$. Thus, in the linear invariant case, the results for convex bodies close to the unit ball in the Banach–Mazur distance immediately follow from those for convex bodies close to the unit ball in the Hausdorff one.

Dealing with problems that are invariant under linear transformations presents one more difficulty: the ball is no longer going to be a unique solution here; any ellipsoid will be just as good.

To avoid this non-uniqueness, in such cases we will always consider the so-called isotropic position of K, that is, the linear image of K for which the quadratic form $x \mapsto \int_K \langle x, y\rangle^2 dy$ is a multiple of $|x|^2$ (see [4, Section 2.3.2] or [1, Section 5] for details). What is important for us here is that if $d(K, B)$ is small, then the Hausdorff distance from the isotropic position of K to B is also small. More precisely, if $d(K, B) < \varepsilon$, i. e., $(1-\varepsilon)B \subset K \subset (1+\varepsilon)B$, then for the isotropic position K' of K, we have

$$(1-\varepsilon)\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{n+2}{2}} B \subset K' \subset (1+\varepsilon)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n+2}{2}} B$$

(see [1, Section 5]).

2.1 Spherical harmonics

We shall now briefly remind the reader of a few basic definitions and facts from harmonic analysis on the unit sphere. More details and applications can be found in [7].

Let $\mathcal{P}$ be the linear space of polynomials of n variables, i. e., finite linear combinations of monomials x^α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{Z}_+$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. If $P = \sum_\alpha c_\alpha x^\alpha \in \mathcal{P}$, then by $P(D)$ we shall mean the differential operator $\sum_\alpha c_\alpha (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$.

Let $\langle P, Q\rangle = P(D)\overline{Q}|_{x=0}$. Then a direct computation shows that $\langle x^\alpha, x^\beta\rangle = 0$ if $\alpha \ne \beta$ and $\langle x^\alpha, x^\alpha\rangle = \alpha! = \alpha_1! \cdots \alpha_n!$. It follows that $\langle\cdot,\cdot\rangle$ is a scalar product on $\mathcal{P}$ for which $\frac{x^\alpha}{\sqrt{\alpha!}}$ is an orthonormal basis.

Let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all homogeneous polynomials of degree m ($m = 0, 1, 2, \ldots$), i. e., the space of linear combinations of monomials x^α with $\alpha_1 + \cdots + \alpha_n = m$. We have $\dim \mathcal{P}_m = \binom{m+n-1}{n-1}$ (see [7, pp. 65–66]). Notice that $|x|^2\mathcal{P}_{m-2}$ is a linear subspace of $\mathcal{P}_m$. If $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_{m-2}$, then

$$\langle |x|^2 Q, P\rangle = (|x|^2 Q)(D)\overline{P}|_{x=0} = Q(D)\overline{\Delta P}|_{x=0} = \langle Q, \Delta P\rangle,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the usual Laplace operator in $\mathbb{R}^n$. Thus, $P \in \mathcal{P}_m$ is orthogonal to $|x|^2\mathcal{P}_{m-2}$ if and only if $\Delta P = 0$, i. e., the space H_m of *harmonic* homogeneous polynomials of degree m is the orthogonal complement of $|x|^2\mathcal{P}_{m-2}$ in $\mathcal{P}_m$. We see that every $P \in \mathcal{P}_m$ can be decomposed as $P_m + |x|^2 Q_{m-2}$, where $P_m \in H_m$ and $Q_{m-2} \in \mathcal{P}_{m-2}$. Repeating this decomposition for Q_{m-2} instead of P and going all the way down, we get the representation

$$P = P_m + |x|^2 P_{m-2} + |x|^4 P_{m-4} + \cdots,$$

where $P_j \in H_j$, $j = m, m-2, m-4, \ldots$. On $\mathbb{S}^{n-1}$, we have $|x|^2 = 1$ and, therefore,

$$P = P_m + P_{m-2} + P_{m-4} + \cdots,$$

i. e., every homogeneous polynomial $P \in \mathcal{P}_m$, as a function on $\mathbb{S}^{n-1}$, can be written as a linear combination of homogeneous harmonic polynomials of degrees $m, m-2, \ldots$. Since every polynomial $P \in \mathcal{P}$ can be decomposed into a sum of homogeneous polynomials, we conclude that every $P \in \mathcal{P}$, as a function on $\mathbb{S}^{n-1}$, can be represented as a sum of finitely many $P_j \in H_j$.

Note that, for $k \neq j$, the Green formula combined with the homogeneity property yields

$$\begin{aligned} 0 = \int_{B_n} [(\Delta P_k)\overline{P}_j - P_k(\Delta \overline{P}_j)]\,dx &= \omega_{n-1} \int_{\mathbb{S}^{n-1}} \left[\left(\frac{\partial}{\partial n} P_k\right)\overline{P}_j - P_k\left(\frac{\partial}{\partial n}\overline{P}_j\right)\right] d\sigma_{n-1} \\ &= \omega_{n-1}(k-j) \int_{\mathbb{S}^{n-1}} P_k \overline{P}_j\, d\sigma_{n-1}, \end{aligned}$$

where ω_{n-1} is the $(n-1)$-dimensional surface area of $\mathbb{S}^{n-1}$ and σ_{n-1} is the normalized surface area measure on $\mathbb{S}^{n-1}$. We see that P_k and P_j are orthogonal with respect to the usual scalar product in $L^2(\mathbb{S}^{n-1})$. Since $\mathcal{P}$ is dense in $L^2(\mathbb{S}^{n-1})$, it follows that we have an orthogonal decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{m=0}^{\infty} H_m,$$

i. e., every function $f \in L^2(\mathbb{S}^{n-1})$ can be uniquely written as a series $f = f_0 + f_1 + f_2 + \cdots$ with $f_m \in H_m$ and the series is orthogonal and convergent in $L^2(\mathbb{S}^{n-1})$. This decomposition is called the spherical harmonic decomposition of f.

2.2 Spherical Radon transform and spherical *k*-Radon transform

Recall that the Fourier transform in $\mathbb{R}^n$ is defined by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle x,y\rangle}\, dx.$$

For reasonable (say, Schwartz class) functions $f : \mathbb{R}^n \to \mathbb{R}^n$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(y) e^{2\pi i \langle x,y\rangle} \, dy.$$

In particular,

$$f(0) = \int_{\mathbb{R}^n} \widehat{f}(y) dy. \tag{2.1}$$

Let $H \subset \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ and let $H^\perp$ be its orthogonal complement. Consider the function

$$F(x') = \int_{x'+H} f(x) dx, \quad x' \in H^\perp.$$

Notice that for $y' \in H^\perp$, we have

$$\int_{H^\perp} F(x') e^{-2\pi i \langle x',y'\rangle} dx' = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,y'\rangle} dx = \widehat{f}(y'),$$

i. e., the Fourier transform of F is just $\widehat{f}|_{H^\perp}$. Applying (2.1), we obtain

$$\int_H f(x) dx = F(0) = \int_{H^\perp} \widehat{F}(y) dy = \int_{H^\perp} \widehat{f}(y) dy. \tag{2.2}$$

Recall also that direct integration by parts yields

$$(P(D)f)^\wedge(y) = \widehat{f}(y) P(2\pi i y)$$

for every polynomial $P \in \mathcal{P}$.

Now take $P \in H_m$ with even $m \geq 0$. A direct computation shows that

$$P(D) e^{-\pi|x|^2} = \big[(2\pi)^m P(-x) + Q(x)\big] e^{-\pi|x|^2} = \big[(2\pi)^m P(x) + Q(x)\big] e^{-\pi|x|^2},$$

where Q is a polynomial of degree $\deg Q < m$. Since $P \in H_m$ is orthogonal to all homogeneous polynomials of degree less than m in $L^2(\mathbb{S}^{n-1})$, it is also orthogonal to them in $L^2(\mathbb{R}^n, w)$ for any fast decaying radial weight w. Hence, it is orthogonal to *all* polynomials of degree less than m in $L^2(\mathbb{R}^n, w)$. In particular,

$$\int_{\mathbb{R}^n} (P(D) e^{-\pi|x|^2}) \overline{Q(x)} dx = \int_{\mathbb{R}^n} |Q(x)|^2 e^{-\pi|x|^2} dx.$$

On the other hand, integration by parts yields

$$\int_{\mathbb{R}^n} (P(D)e^{-\pi|x|^2})\overline{Q(x)}dx = \int_{\mathbb{R}^n} e^{-\pi|x|^2}(P(D)\overline{Q})(x)dx = 0$$

and we conclude that $Q \equiv 0$, so $P(D)e^{-\pi|x|^2} = (2\pi)^m P(x)e^{-\pi|x|^2}$.

Since the Fourier transform of $e^{-\pi|x|^2}$ is $e^{-\pi|y|^2}$, we have

$$(P(D)e^{-\pi|\cdot|^2})^\wedge(y) = P(2\pi iy)e^{-\pi|y|^2} = (-1)^{\frac{m}{2}}(2\pi)^m P(y)e^{-\pi|y|^2}.$$

Thus, (2.2) results in

$$\int_H P(x)e^{-\pi|x|^2}dx = (-1)^{\frac{m}{2}} \int_{H^\perp} P(y)e^{-\pi|y|^2}dy.$$

Using the m-homogeneity of P, we can rewrite this as

$$\begin{aligned}&\int_{H\cap\mathbb{S}^{n-1}} Pd\sigma_{k-1} \times \omega_{k-1} \times \int_0^\infty r^{m+k-1}e^{-\pi r^2}dr\\ &= (-1)^{\frac{m}{2}} \int_{H^\perp\cap\mathbb{S}^{n-1}} Pd\sigma_{n-k-1} \times \omega_{n-k-1} \times \int_0^\infty r^{m+n-k-1}e^{-\pi r^2}dr,\end{aligned}$$

where $k = \dim H$. Taking into account that

$$\int_0^\infty r^{\alpha-1}e^{-\pi r^2}dr = \frac{1}{2}\pi^{-\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right) \quad (\alpha > 0), \quad \omega_{l-1} = \frac{2\pi^{\frac{l}{2}}}{\Gamma(\frac{l}{2})} \quad (l = 1, 2, 3, \dots),$$

we finally get

$$\frac{\Gamma(\frac{m+k}{2})}{\Gamma(\frac{k}{2})} \int_{H\cap\mathbb{S}^{n-1}} Pd\sigma_{k-1} = (-1)^{\frac{m}{2}} \frac{\Gamma(\frac{m+n-k}{2})}{\Gamma(\frac{n-k}{2})} \int_{H^\perp\cap\mathbb{S}^{n-1}} Pd\sigma_{n-k-1},$$

or, equivalently,

$$(-1)^{\frac{m}{2}} \frac{k(k+2)\cdots(k+m-2)}{(n-k)(n-k+2)\cdots(n-k+m-2)} \int_{H\cap\mathbb{S}^{n-1}} Pd\sigma_{k-1} = \int_{H^\perp\cap\mathbb{S}^{n-1}} Pd\sigma_{n-k-1}.$$

When $k = 1$ and H is just a line through a vector $e \in \mathbb{S}^{n-1}$, we get

$$\int_{H^\perp\cap\mathbb{S}^{n-1}} Pd\sigma_{n-2} = (-1)^{\frac{m}{2}} \frac{1 \cdot 3 \cdots (m-1)}{(n-1)(n+1)\cdots(n+m-3)} P(e).$$

It follows that if f is an arbitrary even $L^2(\mathbb{S}^{n-1})$-function and

$$(\mathcal{R}f)(e) = \int\limits_{\mathbb{S}^{n-1}\cap e^{\perp}} f d\sigma_{n-2}$$

is the spherical Radon transform of f, then the spherical harmonic decomposition of $\mathcal{R}f$ is $\mathcal{R}f = \sum_{\substack{m\geq 0\\ m\text{ even}}} c_m f_m$, where $f = \sum_{\substack{m\geq 0\\ m\text{ even}}} f_m$ is the spherical harmonic decomposition of f and

$$c_m = (-1)^{\frac{m}{2}} \frac{1\cdot 3\cdots(m-1)}{(n-1)(n+1)\cdots(n+m-3)}.$$

Observe, in particular, that one can have $\mathcal{R}f \equiv 0$ only if $f \equiv 0$, i. e., the spherical Radon transform $\mathcal{R}f$ of f determines f uniquely.

When $2 \leq k \leq \frac{n}{2}$, we similarly conclude that for every even $L^2(\mathbb{S}^{n-1})$-function $f = \sum_{\substack{m\geq 0\\ m\text{ even}}} f_m$, the function $g = \sum_{\substack{m\geq 0\\ m\text{ even}}} c_{m,k} f_m$, where

$$c_{m,k} = (-1)^{\frac{m}{2}} \frac{k(k+2)\cdots(k+m-2)}{(n-k)(n-k+2)\cdots(n-k+m-2)},$$

is also in $L^2(\mathbb{S}^{n-1})$ and satisfies

$$\int\limits_{H\cap\mathbb{S}^{n-1}} g d\sigma_{k-1} = \int\limits_{H^{\perp}\cap\mathbb{S}^{n-1}} f d\sigma_{n-k-1} \tag{2.3}$$

for every k-dimensional plane $H \subset \mathbb{R}^n$.

Note also that if we know the averages of g over the $(k-1)$-dimensional spheres, we can average further and find the averages over $(n-2)$-dimensional spheres, i. e., we know $\mathcal{R}g$. Thus, the even function g with property (2.3) is unique. We shall call it the spherical k-Radon transform of f and denote $\mathcal{R}_k f$ (so, the usual spherical Radon transform is the same as $\mathcal{R}_1 f$).

At last, observe that the coefficients $c_{m,k}$ make sense and are bounded by 1 in absolute value for every complex k with $0 \leq \operatorname{Re} k \leq \frac{n}{2}$, so we can define $\mathcal{R}_k f$ for such k as well, despite the fact that it has no obvious geometric meaning.

Spherical harmonic decomposition is a powerful tool when we need to prove some result about all convex bodies near the unit ball (and often in a more general case), or, which is almost the same, about all reasonable functions f on $\mathbb{S}^{n-1}$ uniformly close to 1. However, when *constructing counterexamples*, one can often restrict oneself to a much narrower class of convex bodies, the so-called bodies of revolution. They are formally defined as follows. Take any not identically zero concave function $f : [a,b] \to [0,+\infty)$ with $f(a) = f(b) = 0$ and consider

$$K_f = \{x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1} : |x'| \leq f(x_1)\}.$$

The unit ball corresponds to $f_o(t) = \sqrt{1-t^2}$. Note that $f_o'' \leq -1$ on the whole interval $[-1,1]$, so if h is any C^2-function on $[-1,1]$ such that $h(-1) = h(1) = 0$ and $\|h''\|_{C([-1,1])} < 1$, then $f = f_o + h$ is concave and the corresponding body of revolution K_f is convex.

The fact that the whole n-dimensional convex body K_f is completely described by a single real-valued function of one variable whose deviation from f_0 can be varied almost freely often allows one to get away with elementary calculus and ODEs when building convex bodies K close to the unit ball with certain properties.

In what follows, we shall present three "local theorems" and one counterexample illustrating the usage of the above techniques. We tried to choose them so that each one has its own little twist and its own peculiar difficulty that did not appear in the previous ones. We should warn the reader that we sometimes skip the routine details in our presentations and restrict ourselves to the simplest cases of more general theorems. The reader interested in the full exposition and the highest available level of generality should follow the references to the original papers.

3 Example 1: the intersection body problem [8]

Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. The intersection body IK of the body K is the star-shaped body whose radial function is given by

$$\varrho_{IK}(e) = \mathrm{vol}_{n-1}(K \cap e^{\perp}) \quad \forall e \in S^{n-1}.$$

It turns out that, for convex origin-symmetric K, IK is also convex (though we shall not use this fact in any way and the definition makes sense for any star-shaped K). Note that when K is a Euclidean ball centered at the origin, $IK = cK$ for some constant $c > 0$. A question of Lutwak (going back to the late 1980s) is whether there are any other convex, or, more generally, star-shaped bodies with this property in $\mathbb{R}^n$, $n \geq 3$ (in dimension $n = 2$, any body K that is invariant under the rotation by 90° gives an example).

We shall show that the answer is negative if we require in addition that K be close to B. To this end, we shall notice that

$$\mathrm{vol}_{n-1}(K \cap e^{\perp}) = c_n \mathcal{R}[\varrho_K^{n-1}](e) \quad \forall e \in S^{n-1},$$

where $c_n > 0$ is some numerical constant. Note that since σ_{n-2} is normalized by the condition that its total mass is 1, we have $\mathcal{R}1 = 1$.

The Lutwak intersection body problem can be now restated as follows: Does there exist a non-constant positive function f such that

$$\mathcal{R}[f^{n-1}] = cf \tag{3.1}$$

for some $c > 0$?

Note that the property in question is invariant under homotheties, i. e., for every $t > 0$, the condition $\mathcal{R}[f^{n-1}] = cf$ implies

$$\mathcal{R}[(tf)^{n-1}] = t^{n-1}\mathcal{R}[f^{n-1}] = c\,t^{n-2}(tf).$$

Hence, we can always normalize f so that its average over $\mathbb{S}^{n-1}$ equals 1. If f was close to 1, then it will still remain so after this normalization. Note also that if $1-\varepsilon \le f \le 1+\varepsilon$, then $(1-\varepsilon)^{n-1} \le \mathcal{R}[f^{n-1}] \le (1+\varepsilon)^{n-1}$. Now, since $(\mathcal{R}F)(-e) = (\mathcal{R}F)(e)$ for any function F and any $e \in \mathbb{S}^{n-1}$, every solution of our equation $\mathcal{R}[f^{n-1}] = cf$ must be an even function and if $1-\varepsilon \le f \le 1+\varepsilon$, then $(1-\varepsilon)^{n-1}(1+\varepsilon)^{-1} \le c \le (1+\varepsilon)^{n-1}(1-\varepsilon)^{-1}$.

Let now $f = 1 + f_2 + f_4 + \cdots$ be the spherical harmonic decomposition of f. Put $\varphi = f - 1 = f_2 + f_4 + \cdots$. We know that $\|\varphi\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon < 1$. Thus,

$$\left|f^{n-1} - 1 - (n-1)\varphi\right| \le C\varepsilon|\varphi|$$

and

$$|\mathcal{R}[f^{n-1}] - 1 - (n-1)\mathcal{R}\varphi| \le C\varepsilon\mathcal{R}[|\varphi|].$$

On the other hand, as we saw above, $\mathcal{R}\varphi = c_2 f_2 + c_4 f_4 + \cdots$. The coefficients c_m ($m \ge 2$, m even) have been computed in the previous section: they are given by

$$c_m = (-1)^{\frac{m}{2}} \frac{1 \cdot 3 \cdots (m-1)}{(n-1)(n+1)\cdots(n+m-3)}.$$

Thus, $c_2 = -\frac{1}{n-1}$, $|c_m| \le c_4 < \frac{1}{n-1}$ for $m \ge 4$ (we used the fact that $n \ge 3$ to get the last property).

If $\mathcal{R}[f^{n-1}] = cf = c + c\varphi$, we get

$$\left|(c-1) + c\varphi - (n-1)\mathcal{R}\varphi\right| \le C\varepsilon\mathcal{R}[|\varphi|]$$

and, in particular,

$$\left\|(c-1) + c\varphi - (n-1)\mathcal{R}\varphi\right\|_{L^2(\mathbb{S}^{n-1})} \le C\varepsilon\left\|\mathcal{R}[|\varphi|]\right\|_{L^2(\mathbb{S}^{n-1})} \le C\varepsilon\|\varphi\|_{L^2(\mathbb{S}^{n-1})}.$$

Since both φ and $\mathcal{R}[\varphi]$ are orthogonal to constants, we conclude from here that

$$\|c\varphi - (n-1)\mathcal{R}\varphi\|_{L^2(\mathbb{S}^{n-1})} \le C\varepsilon\|\varphi\|_{L^2(\mathbb{S}^{n-1})}$$

as well.

However, the left-hand side squared equals

$$\begin{aligned}\Bigg\|\sum_{\substack{m\ge2\\ m\text{ even}}}(c-(n-1)c_m)f_m\Bigg\|^2_{L^2(\mathbb{S}^{n-1})} &= \sum_{\substack{m\ge2\\ m\text{ even}}}(c-(n-1)c_m)^2\|f_m\|^2_{L^2(\mathbb{S}^{n-1})}\\ &\ge \inf_{\substack{m\ge2\\ m\text{ even}}}(c-(n-1)c_m)^2\sum_{\substack{m\ge2\\ m\text{ even}}}\|f_m\|^2_{L^2(\mathbb{S}^{n-1})}\\ &= \inf_{\substack{m\ge2\\ m\text{ even}}}(c-(n-1)c_m)^2\|\varphi\|^2_{L^2(\mathbb{S}^{n-1})}.\end{aligned}$$

It remains to note that

$$c - (n-1)c_2 = c + 1 > 1,$$

while for $m \geq 4$,

$$\left|c - (n-1)c_m\right| \geq 1 - (n-1)|c_4| - |c-1|,$$

so if ε is so small that

$$C\varepsilon + |c-1| < 1 - (n-1)|c_4|,$$

we get a contradiction unless $\varphi \equiv 0$.

This simple argument illustrates the main advantage of the perturbative regime: a possibility to switch from a non-linear equation (or inequality) to its linearization. This is a trick we shall be using again and again.

It is natural to ask now what happens if we replace the intersection body operator with some of its iterations, for example, if we ask when

$$I^2K = I(IK) = cK,$$

i. e., consider the equation

$$\mathcal{R}\left[\left(\mathcal{R}[f^{n-1}]\right)^{n-1}\right] = cf.$$

If we try to treat this equation in the same way as the previous one, after linearizing, we shall arrive at the inequality

$$\left\|(c-1) + c\varphi - (n-1)^2\mathcal{R}^2[\varphi]\right\|_{L^2(\mathbb{S}^{n-1})} \leq C\varepsilon\|\varphi\|_{L^2(\mathbb{S}^{n-1})}.$$

We can again remove $c-1$ on the left-hand side and write

$$c\varphi - (n-1)^2\mathcal{R}^2[\varphi] = \sum_{\substack{m\geq 2\\ m\text{ even}}} (c - (n-1)^2c_m^2)f_m.$$

For $m \geq 4$, we still have $c - (n-1)^2c_m^2$ separated from 0 if $\varepsilon > 0$ is small enough. However, $(n-1)^2c_2^2 = 1$ now and the coefficient at f_2 can be arbitrarily small, so the previous argument fails. This is not accidental: unlike the equation $IK = cK$, which was invariant only under homotheties, the equation $I^2K = cK$ is invariant under arbitrary linear transformations (see [6, Theorem 8.1.16]).

Therefore, any ellipsoid gives a solution and it is no longer possible to conclude that $\varphi \equiv 0$ from the equation above. Fortunately, the very same invariance under linear transformations allows us to put the body K into the isotropic position. Then, for every quadratic form $Q(e) = \sum_{i,j} a_{ij}e_ie_j$ with $\sum_i a_{ii} = 0$, we have

$$\int_{\mathbb{S}^{n-1}} \varrho_K(e)^{n+2} Q(e) d\sigma_{n-1}(e) = c_n \int_K Q(x)dx = 0,$$

which means that for the function $f = \varrho_K$, f^{n+2} is orthogonal to all spherical harmonics of degree 2. Representing $f = 1 + \varphi$, using the bound $\|\varphi\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon$, and linearizing, we get

$$\left|f^{n+2} - 1 - (n+2)\varphi\right| \leq C\varepsilon|\varphi|.$$

We see that

$$\left\|f^{n+2} - 1 - (n+2)\varphi\right\|_{L^2(\mathbb{S}^{n-1})} \leq C\varepsilon\|\varphi\|_{L^2(\mathbb{S}^{n-1})}.$$

However, the second-order spherical harmonic in the expansion of $f^{n+2} - 1 - (n+2)\varphi$ is just $-(n+2)f_2$, so we conclude that

$$\|f_2\|_{L^2(\mathbb{S}^{n-1})} \leq C\varepsilon\|\varphi\|_{L^2(\mathbb{S}^{n-1})},$$

and, for sufficiently small $\varepsilon > 0$, the L^2-norm of φ comes mainly from the spherical harmonics f_m with $m \geq 4$, i. e., we can write

$$\left\|c\varphi - (n-1)^2\mathcal{R}^2[\varphi]\right\|^2_{L^2(\mathbb{S}^{n-1})} \geq \gamma^2 \sum_{\substack{m\geq 4\\ m\,\text{even}}} \|f_m\|^2_{L^2(\mathbb{S}^{n-1})} \geq \frac{\gamma^2}{2}\|\varphi\|^2_{L^2(\mathbb{S}^{n-1})}$$

with

$$\gamma = \inf_{\substack{m\geq 4\\ m\,\text{even}}} \left|c - (n-1)^2 c_m^2\right| \geq 1 - (n-1)^2 c_4^2 - |c-1|,$$

so if ε is so small that

$$C\varepsilon + |c-1| < 1 - (n-1)^2 c_4^2,$$

we get a contradiction unless $\varphi \equiv 0$.

The reader should by now be able to show that for any $k \geq 1$, the only star-shaped bodies close to the unit ball that satisfy the equation $I^k K = cK$ are balls if k is odd and ellipsoids if k is even. The full result of [8] is stronger. Namely, it is shown there that if the body K is sufficiently close to the ball, then the iterations $I^k K$ converge to the unit ball in the Banach–Mazur distance, so IK (or $I^k K$) can be neither a homothetic image of K nor even a linear image of K, unless K is an ellipsoid. The proof of this stronger statement is more complicated, so we refer the reader to the original paper for details.

4 Example 2: Busemann's inequality [15]

The well-known Busemann intersection inequality asserts that for any star-shaped body K, we have

$$\mathrm{vol}_n(IK) \le \frac{\kappa_{n-1}^n}{\kappa_n^{n-2}} \mathrm{vol}_n(K)^{n-1}$$

with equality attained if and only if K is a centered ellipsoid (see [6, Corollary 9.4.5]). Here $\kappa_p = \frac{\pi^{\frac{p}{2}}}{\Gamma(1+\frac{p}{2})}$, which equals the volume of the unit ball in $\mathbb{R}^p$ when p is a positive integer.

Koldobsky introduced a generalization of the notion of an intersection body (see [9, p. 75]). Let K and L be origin-symmetric star-shaped bodies in $\mathbb{R}^n$ and let k be an integer, $1 \le k \le n-1$. We say that L is the k-intersection body of K if

$$\mathrm{vol}_k(L \cap H) = \mathrm{vol}_{n-k}(K \cap H^{\perp})$$

for every k-dimensional subspace H of $\mathbb{R}^n$. Note that the 1-intersection body of K is $\frac{1}{2}IK$.

It is worth mentioning that when $k > 1$, for a given origin-symmetric star-shaped body K, its k-intersection body may not exist in general.

It has been conjectured (see [10]) that an analog of Busemann's intersection inequality holds for k-intersection bodies when $k \le \frac{n}{2}$, i. e., if L is a k-intersection body of K, then

$$\mathrm{vol}_n(L)^k \le C_{n,k} \,\mathrm{vol}_n(K)^{n-k}, \tag{4.1}$$

where $C_{n,k} > 0$ is the constant that turns (4.1) into an equality when K is a ball.

The condition that L is the k-intersection body of K can be analytically expressed in terms of the spherical k-Radon transform as

$$\varrho_L^k = b_{n,k}\mathcal{R}_k[\varrho_K^{n-k}],$$

where $b_{n,k} > 0$ is some numerical coefficient. This relation allows one to rewrite inequality (4.1) as

$$\Big(\int_{\mathbb{S}^{n-1}} \mathbb{R}_k[\varrho_K^{n-k}]^{\frac{n}{k}}\, d\sigma_{n-1}\Big)^k \le \Big(\int_{\mathbb{S}^{n-1}} \varrho_K^n d\sigma_{n-1}\Big)^{n-k} = \Big(\int_{\mathbb{S}^{n-1}} (\varrho_K^{n-k})^{\frac{n}{n-k}}\, d\sigma_{n-1}\Big)^{n-k} \tag{4.2}$$

(here one clearly has equality for $\varrho_K \equiv \mathit{const}$).

Raising both sides to the power $\frac{1}{n}$, we see that this inequality is almost the same as the statement that the operator norm

$$\|\mathcal{R}_k\|_{L^{\frac{n}{n-k}}(\mathbb{S}^{n-1}) \to L^{\frac{n}{k}}(\mathbb{S}^{n-1})}$$

is at most 1, except we can additionally assume that both the test function and its image are even and positive.

If $k = \frac{n}{2}$, then we have $|c_{m,k}| = 1$ for all even $m \geq 2$, so $\mathcal{R}_k$ is an isometry in $L^2_{\text{even}}(\mathbb{S}^{n-1})$ and there is nothing to prove. Assume now that $k < \frac{n}{2}$. In this case we shall prove only that the desired inequality holds if ϱ_K is close to 1, i. e., K is close to the unit ball.

The geometric meaning of the problem is useful in one more respect: it allows one to observe that inequality (4.2) is invariant under linear transformations of K (see [15]). Thus, we can assume that K is in the isotropic position, i. e., the function $\varrho_K^{n+2} = (\varrho_K^{n-k})^{\frac{n+2}{n-k}}$ has no second-order spherical harmonics. We can also assume that $\int_{S^{n-1}} \varrho_K^{n-k} d\sigma_{n-1} = 1$. Then, denoting $\varrho_K^{n-k} = f = 1 + \varphi$ and arguing as in the previous example, we see that if K is close enough to the unit ball, i. e., $\|\varphi\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon$ with sufficiently small ε, then the contribution of f_2 into the $L^2(\mathbb{S}^{n-1})$-norm of φ is negligible, so

$$\sum_{\substack{m \geq 4 \\ m \text{ even}}} \|f_m\|^2_{L^2(\mathbb{S}^{n-1})} \geq \frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{S}^{n-1})},$$

say.

We thus want to show that

$$\int_{\mathbb{S}^{n-1}} (1 + \mathcal{R}_k[\varphi])^{\frac{n}{k}} d\sigma_{n-1} \leq \left(\int_{\mathbb{S}^{n-1}} (1+\varphi)^{\frac{n}{n-k}} d\sigma_{n-1} \right)^{\frac{n-k}{k}}. \tag{4.3}$$

It is tempting to expand to the second order and write the left- and right-hand sides of (4.3) as

$$\begin{aligned} LHS &\approx \int_{\mathbb{S}^{n-1}} \left(1 + \frac{n}{k}\mathcal{R}_k[\varphi] + \frac{1}{2}\frac{n}{k}\left(\frac{n}{k} - 1\right)(\mathcal{R}_k[\varphi])^2\right) d\sigma_{n-1} \\ &= 1 + \frac{1}{2}\frac{n}{k}\left(\frac{n}{k} - 1\right) \int_{\mathbb{S}^{n-1}} (\mathcal{R}_k[\varphi])^2 d\sigma_{n-1} \end{aligned}$$

and

$$\begin{aligned} RHS &\approx \left(\int_{\mathbb{S}^{n-1}} \left(1 + \frac{n}{n-k}\varphi + \frac{1}{2}\frac{n}{n-k}\left(\frac{n}{n-k} - 1\right)\varphi^2\right) d\sigma_{n-1} \right)^{\frac{n-k}{k}} \\ &= \left(1 + \frac{1}{2}\frac{n}{n-k}\left(\frac{n}{n-k} - 1\right) \int_{\mathbb{S}^{n-1}} \varphi^2 d\sigma_{n-1}\right)^{\frac{n-k}{k}} \\ &\approx 1 + \frac{1}{2}\frac{n}{k}\left(\frac{n}{n-k} - 1\right) \int_{\mathbb{S}^{n-1}} \varphi^2 d\sigma_{n-1}. \end{aligned}$$

Then it would remain to conclude that

$$RHS - LHS \approx \frac{1}{2}\frac{n}{k}\left(\frac{k}{n-k} \int_{\mathbb{S}^{n-1}} \varphi^2 \, d\sigma_{n-1} - \frac{n-k}{k} \int_{\mathbb{S}^{n-1}} (\mathcal{R}_k[\varphi])^2 d\sigma_{n-1}\right)$$

$$= \frac{1}{2}\frac{n}{k}\sum_{\substack{m\geq 4\\ m\,\text{even}}}\left(\frac{k}{n-k} - c_{m,k}^2\frac{n-k}{k}\right)\int_{\mathbb{S}^{n-1}} f_m^2\,d\sigma_{n-1}$$

and observe that $c_{2,k} = -\frac{k}{n-k}$ and $|c_{m,k}| \leq |c_{4,k}| < \frac{k}{n-k}$ for even $m \geq 4$, so the expression in the final line of the last formula is at least

$$\gamma \sum_{\substack{m\geq 4\\ m\,\text{even}}} \int_{\mathbb{S}^{n-1}} f_m^2\,d\sigma_{n-1} \geq \frac{\gamma}{2}\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1}$$

with some $\gamma = \gamma(k,n) > 0$.

To justify this approach, one needs, however, to show that the errors in the second-order Taylor approximations are small compared to $\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1}$ when ε is close to 0.

The right-hand side *RHS* presents no problem because we control φ in $L^\infty(\mathbb{S}^{n-1})$. The main difficulty with the left-hand side *LHS* is that unlike the usual Radon transform, $\mathcal{R}_k$ is not bounded in $L^\infty(\mathbb{S}^{n-1})$ for $k > 1$ and we cannot say that $\mathcal{R}_k[\varphi]$ is small at each individual point. What we shall use instead is the fact that $\mathcal{R}_k$ is bounded from $L^2(\mathbb{S}^{n-1})$ to $L^{\frac{n}{k}}(\mathbb{S}^{n-1})$ for all $0 \leq k \leq \frac{n}{2}$. Assuming that, we will use the inequality

$$\left||1+t|^p - \left(1 + pt + \frac{p(p-1)}{2}t^2\right)\right| \leq \begin{cases} c_p|t|^p, & 2 < p \leq 3,\\ c_p(|t|^p + |t|^3), & p \geq 3,\end{cases}$$

valid for all $t \in \mathbb{R}$ and $p > 2$. To prove it, just notice that the ratio of its left-hand side to $|t|^p$ or $|t|^p + |t|^3$, respectively, is a continuous function on $\mathbb{R} \setminus \{0\}$ that stays bounded as $t \to 0$ and as $t \to \pm\infty$.

This inequality allows one to estimate the error in the approximation of $\int_{\mathbb{S}^{n-1}}(1 + \mathcal{R}_k[\varphi])^{\frac{n}{k}}\,d\sigma_{n-1}$ by a constant multiple of $\int_{\mathbb{S}^{n-1}} |\mathcal{R}_k[\varphi]|^{\frac{n}{k}}\,d\sigma_{n-1}$ if $2 < \frac{n}{k} \leq 3$ and of $\int_{\mathbb{S}^{n-1}}(|\mathcal{R}_k[\varphi]|^{\frac{n}{k}} + |\mathcal{R}_k[\varphi]|^3)d\sigma_{n-1}$ if $\frac{n}{k} > 3$.

Using the boundedness of $\mathcal{R}_k$ from $L^2(\mathbb{S}^{n-1})$ to $L^{\frac{n}{k}}(\mathbb{S}^{n-1})$, we immediately get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \left|\mathcal{R}_k[\varphi]\right|^{\frac{n}{k}} d\sigma_{n-1} &= \left\|\mathcal{R}_k[\varphi]\right\|^{\frac{n}{k}}_{L^{\frac{n}{k}}(\mathbb{S}^{n-1})} \leq C\|\varphi\|^{\frac{n}{k}}_{L^2(\mathbb{S}^{n-1})}\\ &= C\Big(\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1}\Big)^{\frac{n}{2k}} \leq C\varepsilon^{\frac{n}{k}-2}\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1},\end{aligned}$$

and if $\frac{n}{k} > 3$, we also have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \left|\mathcal{R}_k[\varphi]\right|^3 d\sigma_{n-1} &= \left\|\mathcal{R}_k[\varphi]\right\|^3_{L^3(\mathbb{S}^{n-1})} \leq \left\|\mathcal{R}_k[\varphi]\right\|^3_{L^{\frac{n}{k}}(\mathbb{S}^{n-1})}\\ &\leq C\|\varphi\|^3_{L^2(\mathbb{S}^{n-1})} = C\Big(\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1}\Big)^{\frac{3}{2}} \leq C\varepsilon\int_{\mathbb{S}^{n-1}} \varphi^2\,d\sigma_{n-1}.\end{aligned}$$

It remains only to prove the boundedness of $\mathcal{R}_k$ from $L^2(\mathbb{S}^{n-1})$ to $L^{\frac{n}{k}}(\mathbb{S}^{n-1})$ for $0 \leq k \leq \frac{n}{2}$.

Recall that the definition

$$c_{m,k} = (-1)^{\frac{m}{2}} \frac{k(k+2)\cdots(k+m-2)}{(n-k)(n-k+2)\cdots(n-k+m-2)}$$

makes sense not only for integer k but for any $k = z \in \mathbb{C}$ with $0 \le \operatorname{Re} z \le \frac{n}{2}$. Moreover, for every such z, we have $|c_{m,z}| \le 1$, so the mapping R_z defined by

$$f = f_0 + f_2 + f_4 + \cdots \quad \mapsto \quad R_z f = c_{0,z} f_0 + c_{2,z} f_2 + c_{4,z} f_4 + \cdots$$

for any even $L^2(\mathbb{S}^{n-1})$-function f is bounded in $L^2_{\text{even}}(\mathbb{S}^{n-1})$ with

$$\|R_z\|_{L^2_{\text{even}}(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})} \le 1$$

and depends on z analytically in the strip $0 < \operatorname{Re} z < \frac{n}{2}$. Moreover, for every fixed $f \in L^2_{\text{even}}(\mathbb{S}^{n-1})$, the mapping $z \mapsto R_z f$ is continuous up to the boundary of this strip.

By the Stein interpolation theorem (see [13] or [14, Chapter 5]), it now suffices to show that when $\operatorname{Re} z = 0$, $\|R_z\|_{L^2_{\text{even}}(\mathbb{S}^{n-1}) \to L^\infty(\mathbb{S}^{n-1})}$ is finite and grows at most polynomially as $|z| \to \infty$. To this end, we first estimate $\|f_m\|_{L^\infty(\mathbb{S}^{n-1})}$ in terms of $\|f_m\|_{L^2(\mathbb{S}^{n-1})}$.

Observe that H_m is a finite-dimensional linear space of harmonic polynomials. Let $g_1, \dots, g_N$ be an orthonormal basis in H_m with respect to the scalar product in $L^2(\mathbb{S}^{n-1})$. Then, for every $g \in H_m$, we have $g = \sum_{j=1}^N \langle g, g_j\rangle g_j$, so for every $x \in \mathbb{S}^{n-1}$ we have

$$g(x) = \left\langle g, \sum_{j=1}^N \overline{g_j(x)} g_j \right\rangle = \langle g, \mathcal{G}_x\rangle.$$

Thus, the norm of the linear functional $g \mapsto g(x)$ on H_m equals

$$\|\mathcal{G}_x\|_{L^2(\mathbb{S}^{n-1})} = \left\| \sum_{j=1}^N \overline{g_j(x)} g_j \right\|_{L^2(\mathbb{S}^{n-1})} = \left(\sum_{j=1}^N |g_j(x)|^2 \right)^{\frac{1}{2}}.$$

On the other hand, the space H_m is invariant under rotations, so this norm must be independent of x. Therefore, for every $x \in \mathbb{S}^{n-1}$, we have

$$\|\mathcal{G}_x\|^2_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} \|\mathcal{G}_x\|^2_{L^2(\mathbb{S}^{n-1})} d\sigma_{n-1}(x) = \int_{\mathbb{S}^{n-1}} \sum_{j=1}^N |g_j(x)|^2 d\sigma_{n-1}(x) = N.$$

Recall now that

$$\dim H_m = \dim \mathcal{P}_m - \dim \mathcal{P}_{m-2} = \binom{m+n-1}{n-1} - \binom{m-2+n-1}{n-1} = D_m = O(m^{n-2})$$

as $m \to \infty$. We see that

$$\|f_m\|_{L^\infty(\mathbb{S}^{n-1})} \le \sqrt{D_m} \|f_m\|_{L^2(\mathbb{S}^{n-1})}$$

and

$$\begin{aligned}
\|R_z f\|_{L^\infty(\mathbb{S}^{n-1})} &\le \sum_{\substack{m\ge 0\\ m\text{ even}}} |c_{m,z}| \|f_m\|_{L^\infty(\mathbb{S}^{n-1})} \\
&\le \sum_{\substack{m\ge 0\\ m\text{ even}}} |c_{m,z}| \sqrt{D_m} \|f_m\|_{L^2(\mathbb{S}^{n-1})} \\
&\le \Bigg(\sum_{\substack{m\ge 0\\ m\text{ even}}} |c_{m,z}|^2 D_m\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m\ge 0\\ m\text{ even}}} \|f_m\|^2_{L^2(\mathbb{S}^{n-1})}\Bigg)^{\frac{1}{2}} \\
&= \Bigg(\sum_{\substack{m\ge 0\\ m\text{ even}}} |c_{m,z}|^2 D_m\Bigg)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{S}^{n-1})}.
\end{aligned}$$

It remains to estimate $|c_{m,z}|$. When n is even, it is very easy. Since $\operatorname{Re} z = 0$, $|n - z + j| = |z + n + j|$ for every $j = 0, 2, \dots, m-2$, so

$$|c_{m,z}| = \frac{\prod_{\substack{0\le j\le m-2\\ j\text{ even}}} |z+j|}{\prod_{\substack{0\le j\le m-2\\ j\text{ even}}} |z+n+j|} = \frac{\prod_{\substack{0\le j\le n-2\\ j\text{ even}}} |z+j|}{\prod_{\substack{m\le j\le m+n-2\\ j\text{ even}}} |z+j|} \le \frac{|z||z+2|\cdots|z+n-2|}{m^{\frac{n}{2}}}$$

for $m \ge 2$. We also have $|c_{0,z}| = 1$. Thus,

$$\sum_{\substack{m\ge 0\\ m\text{ even}}} |c_{m,z}|^2 D_m \le 1 + |z|^2|z+2|^2\cdots|z+n-2|^2 \sum_{\substack{m\ge 2\\ m\text{ even}}} \frac{D_m}{m^n}.$$

Since $D_m = O(m^{n-2})$, the series on the right-hand side converges and we are done. A similar estimate holds for odd n too but its proof is a bit more complicated, so we refer the reader to [15] for details.

5 Example 3: a local version of the fifth Busemann–Petty problem [1]

Busemann and Petty [5] asked the following question. Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Suppose that for every $(n-1)$-dimensional subspace $H \subset \mathbb{R}^n$, the volume of the cone of the largest volume with base $K \cap H$ contained in K does not depend on H. Does it follow that K is an ellipsoid?

The equivalent analytic reformulation of the question assumption is that the product $\mathfrak{h}_K(e)\mathcal{R}[\varrho_K^{n-1}](e) = const$ on $\mathbb{S}^{n-1}$, or, equivalently,

$$\mathfrak{h}_K(e) = c\big(\mathcal{R}[\varrho_K^{n-1}](e)\big)^{-1}. \tag{5.1}$$

The formulation of the problem is, clearly, invariant under linear transformations, so we can assume from the beginning that the body K is in the isotropic position.

We shall also normalize the convex body K by the condition

$$\int_{\mathbb{S}^{n-1}} \varrho_K d\sigma_{n-1} = 1. \tag{5.2}$$

Our task will be to show that if, under such conditions, K is close to the unit ball B, then $K = B$.

Equation (5.1) should remind the reader of equation (3.1) in the intersection body example. Just as in that example, one can conclude that since both $\mathfrak{h}_K$ and ϱ_K are close to 1, the constant c must be close to 1 as well. The key difference and the main difficulty, however, is that now we have two different functions $\mathfrak{h}_K$ and ϱ_K in the equation and while each of them determines the other one uniquely, the relation between them is highly non-linear.

What saves the day is that this non-linearity manifests itself only on high frequencies. More precisely, we have the following.

Lemma. *Let $\varrho = \varrho_K$ and $\mathfrak{h} = \mathfrak{h}_K$ be the radial and support functions of an origin-symmetric convex body K close to the unit ball, respectively. Let $\mathfrak{h} = \sum_{\substack{m\geq 0\\ m \text{ even}}} \mathfrak{h}_m$ be the spherical harmonic decomposition of $\mathfrak{h}$. Put $\eta = \sum_{\substack{2\leq m\leq l\\ m \text{ even}}} \mathfrak{h}_m$, $\nu = \sum_{\substack{m>l\\ m \text{ even}}} \mathfrak{h}_m$. Then for every $\varepsilon, l > 0$, there exists $\delta_o = \delta_o(\varepsilon, l)$ such that whenever $\|\mathfrak{h} - 1\|_\infty \leq \delta_o$, the inequality*

$$0 \leq \mathfrak{h} - \varrho \leq \varepsilon \|\eta\|_{L^2(\mathbb{S}^{n-1})} + CM\nu$$

holds, where

$$M\nu(e) = \max_{\theta\in(0,\pi)} \frac{1}{\sigma_{n-1}(S_\theta(e))} \int_{S_\theta(e)} |\nu(x)| d\sigma_{n-1}(x)$$

is the spherical Hardy–Littlewood maximal function, $S_\theta(e)$ denotes the set of vectors $x \in \mathbb{S}^{n-1}$ making an angle less than θ with the vector $e \in \mathbb{S}^{n-1}$, and $C > 0$ is a constant depending only on the dimension n.

Proof. We have

$$\varrho(e) = \inf\left\{ \frac{\mathfrak{h}(e')}{\langle e, e'\rangle} : e' \in \mathbb{S}^{n-1}, \langle e, e'\rangle > 0 \right\}.$$

Note that the admissible range of e' can be further restricted to $|e - e'| < \delta$ with arbitrarily small $\delta > 0$, provided that δ_o is chosen small enough. Indeed, since $\mathfrak{h}(e') \geq \frac{1-\delta_o}{1+\delta_o}\mathfrak{h}(e)$, e' can compete with e only if $\langle e, e'\rangle \geq \frac{1-\delta_o}{1+\delta_o}$, so

$$|e - e'|^2 = 2(1 - \langle e, e'\rangle) \leq \frac{4\delta_o}{1+\delta_o} < \delta^2$$

if $\delta_o > 0$ is chosen appropriately. Now observe also that all norms on the finite-dimensional space of polynomials of degree not exceeding l on the unit sphere are equivalent and that any seminorm is dominated by any norm, whence

$$\|\eta\|_{C(\mathbb{S}^{n-1})} \le C(l)\|\eta\|_{L^2(\mathbb{S}^{n-1})} \quad \text{and} \quad \|\nabla\eta\|_{C(\mathbb{S}^{n-1})} \le C(l)\|\eta\|_{L^2(\mathbb{S}^{n-1})}.$$

In particular, if $|e - e''| < 2\delta$, we get

$$\left|\eta(e) - \eta(e'')\right| \le 4\|\nabla\eta\|_{C(\mathbb{S}^{n-1})}\delta \le 4C(l)\delta\|\eta\|_{L^2(\mathbb{S}^{n-1})}.$$

Before we proceed, let us prove the following claim. Let $R > \omega > 0$ and let $e \in \mathbb{S}^{n-1}$ be a unit vector. Assume that $\mathfrak{h}(e) = (R-\omega)\cos\theta$ for some $\theta \in (0, \frac{\pi}{3})$. Then

$$\frac{1}{\sigma(S_\theta(e))} \int\limits_{S_\theta(e)} \left|\mathfrak{h}(e') - R\right| d\sigma_{n-1}(e') \ge c\,\omega \tag{5.3}$$

with some $c > 0$ depending on n only.

We will use the parametrization $e' = e'(t, v) = e\cos t + v\sin t$, where $t < \theta$ is the angle between e and e' and $v \in \mathbb{S}^{n-1} \cap e^\perp$ is the direction of the projection of e' to $e^\perp$. Note that

$$d\sigma_{n-1}(e') = c_n(\sin t)^{n-2}\, dt\, d\sigma_{n-2}(v).$$

Since $e'(\frac{t}{2}, v)\cos\frac{t}{2} = \frac{1}{2}(e'(t,v) + e)$, by the convexity and 1-homogeneity of $\mathfrak{h}$ we have

$$\mathfrak{h}\left(e'\left(\frac{t}{2}, v\right)\right)\cos\frac{t}{2} \le \frac{1}{2}[\mathfrak{h}(e'(t,v)) + \mathfrak{h}(e)],$$

so

$$\begin{aligned} 2\mathfrak{h}\left(e'\left(\frac{t}{2}, v\right)\right)\cos\frac{t}{2} - \mathfrak{h}(e'(t,v)) &\le \mathfrak{h}(e) \\ &= (R-\omega)\cos\theta = R\left(2\cos^2\frac{\theta}{2} - 1\right) - \omega\cos\theta \le R\left(2\cos\frac{t}{2} - 1\right) - \frac{\omega}{2}, \end{aligned}$$

whence

$$\begin{aligned} 2&\left|\mathfrak{h}\left(e'\left(\frac{t}{2}, v\right)\right) - R\right| + \left|\mathfrak{h}(e'(t,v)) - R\right| \\ &\ge 2\left|\mathfrak{h}\left(e'\left(\frac{t}{2}, v\right)\right) - R\right|\cos\frac{t}{2} + \left|\mathfrak{h}(e'(t,v)) - R\right| \ge \frac{\omega}{2}. \end{aligned}$$

Integrating this inequality against

$$c_n(\sin t)^{n-2}\,dt\,d\sigma_{n-2}(v) \le 2^{n-1}c_n\left(\sin\frac{t}{2}\right)^{n-2} d\left(\frac{t}{2}\right) d\sigma_{n-2}(v),$$

we get

$$2^n \int\limits_{S_{\frac{\theta}{2}}(e)} |\mathfrak{h}(e') - R|d\sigma_{n-1}(e') + \int\limits_{S_\theta(e)} |\mathfrak{h}(e') - R|d\sigma_{n-1}(e') \ge \frac{\omega}{2}\sigma_{n-1}(S_\theta(e))$$

and the desired inequality follows with $c = \frac{1}{2(2^n+1)}$.

Let us now assume that $e' \in \mathbb{S}^{n-1}$ with $|e - e'| < \delta$ is such that

$$\frac{\mathfrak{h}(e')}{\langle e, e'\rangle} = \varrho(e) < \mathfrak{h}(e).$$

Then, if θ is the angle between e and e', we have

$$\mathfrak{h}(e') = [\mathfrak{h}(e) - (\mathfrak{h}(e) - \varrho(e))]\cos\theta$$

and we can apply (5.3) to the vector e' with $R = \mathfrak{h}(e)$ and $\omega = \mathfrak{h}(e) - \varrho(e)$ to conclude that

$$\begin{aligned}
\mathfrak{h}(e) - \varrho(e) &\le \frac{C}{\sigma_{n-1}(S_\theta(e'))} \int\limits_{S_\theta(e')} |\mathfrak{h}(e) - \mathfrak{h}(e'')|d\sigma_{n-1}(e'') \\
&\le \frac{C'}{\sigma_{n-1}(S_{2\theta}(e))} \int\limits_{S_{2\theta}(e)} |\mathfrak{h}(e) - \mathfrak{h}(e'')|d\sigma_{n-1}(e'').
\end{aligned}$$

However,

$$|\mathfrak{h}(e) - \mathfrak{h}(e'')| \le |\eta(e) - \eta(e'')| + |\nu(e)| + |\nu(e'')|$$

and

$$|\eta(e) - \eta(e'')| \le 4C(l)\delta\|\eta\|_{L^2(\mathbb{S}^{n-1})},$$

while

$$|\nu(e)| \le M\nu(e) \quad \text{and} \quad \frac{1}{\sigma_{n-1}(S_{2\theta}(e))} \int\limits_{S_{2\theta}(e)} |\nu(e'')|d\sigma_{n-1}(e'') \le M\nu(e),$$

so the desired statement follows if we choose $\delta > 0$ so that $4C'C(l)\delta < \varepsilon$. □

Now fix $\varepsilon > 0$, $l > 0$ to be chosen later and assume that $\|\mathfrak{h} - 1\|_\infty < \delta_o$, where $\delta_o > 0$ is very small. Since the body K is assumed to be in the isotropic position and normalized by (5.2), arguing as in Section 3 we can again show that the second-order spherical harmonics component of ϱ is small compared to $\varrho - 1$. Linearizing the right-hand side of the equation $\mathfrak{h} = c(\mathcal{R}[\varrho^{n-1}])^{-1}$ around 1, we get $\mathfrak{h} = c(1 - (n-1)\mathcal{R}(\varrho - 1) + \gamma)$, where $\|\gamma\|_{L^2(\mathbb{S}^{n-1})} \le \varepsilon \|\varrho - 1\|_{L^2(\mathbb{S}^{n-1})}$, provided that δ_o is small enough.

Since, for small enough δ_o, the $L^2(\mathbb{S}^{n-1})$-norm of the second-order spherical harmonics component of $\varrho - 1$ is also less than $\varepsilon\|\varrho - 1\|_{L^2(\mathbb{S}^{n-1})}$ and $c < 1 + \varepsilon$, we can incorporate the second-order spherical harmonics component into the error term γ and conclude that

$$\big\|\mathfrak{h} - c - c\,\mathfrak{M}(\varrho - 1)\big\|_{L^2(\mathbb{S}^{n-1})} \le 3\varepsilon\|\varrho - 1\|_{L^2(\mathbb{S}^{n-1})}. \tag{5.4}$$

Here $\mathfrak{M}$ is the linear operator that maps every m-th-order spherical harmonic Z_m to $\mu_m Z_m$, where

$$\mu_m = -(n-1)(-1)^{\frac{m}{2}} \frac{1 \cdot 3 \cdots (m-1)}{(n-1)(n+1)\cdots(n+m-3)}$$

for even $m \ge 4$ and $\mu_m = 0$ for other m (so $\mu_m Z_m = -(n-1)\mathcal{R}Z_m$ for even $m \ge 4$). Note that when $n \ge 3$, we have $|\mu_m| < 1$ for all m and $\mu_m \to 0$ as $m \to \infty$.

Consider the decomposition $\mathfrak{h} = \mathfrak{h}_o + \eta + \nu$ and $\varrho = 1 + \varphi + \psi$, where $\mathfrak{h}_o$ is the constant term, η and φ are the parts corresponding to the harmonics of degrees 2 to l, and ν, ψ are the parts corresponding to harmonics of degrees greater than l. Since the projection to any sum of spaces of spherical harmonics in $L^2(\mathbb{S}^{n-1})$ has norm 1, inequality (5.4) implies

$$\|\eta - c\,\mathfrak{M}\varphi\|_{L^2(\mathbb{S}^{n-1})} \le 3\varepsilon\|\varrho - 1\|_{L^2(\mathbb{S}^{n-1})} \le 3\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big)$$

and the same estimate holds for $\|\nu - c\,\mathfrak{M}\psi\|_{L^2(\mathbb{S}^{n-1})}$. Thus,

$$\begin{aligned}
\|\nu\|_{L^2(\mathbb{S}^{n-1})} &\le c\|\mathfrak{M}\psi\|_{L^2(\mathbb{S}^{n-1})} + 3\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big) \\
&\le c \max_{m>l} |\mu_m|\, \|\psi\|_{L^2(\mathbb{S}^{n-1})} + 3\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big) \\
&\le 4\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big),
\end{aligned}$$

provided l is chosen so large that $c\max_{m>l}|\mu_m| < \varepsilon$ and $\delta_o > 0$ is small enough.

The same computation for η, using just the crude bound $\max_{m>0}|\mu_m| < 1$, yields

$$\|\eta\|_{L^2(\mathbb{S}^{n-1})} \le 2\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big).$$

On the other hand, by Lemma 5 and the boundedness of the maximal function in $L^2(\mathbb{S}^{n-1})$, we have

$$\|\mathfrak{h} - \varrho\|_{L^2(\mathbb{S}^{n-1})} \le \varepsilon\|\eta\|_{L^2(\mathbb{S}^{n-1})} + C\|\nu\|_{L^2(\mathbb{S}^{n-1})} \le (2 + 4C)\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big),$$

which implies the same bound for both $\|\varphi - \eta\|_{L^2(\mathbb{S}^{n-1})}$ and $\|\psi - \nu\|_{L^2(\mathbb{S}^{n-1})}$. Combining all the above estimates, we get

$$\|\varphi - c\,\mathfrak{M}\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi - c\,\mathfrak{M}\psi\|_{L^2(\mathbb{S}^{n-1})}$$

$$\le \|\varphi - \eta\|_{L^2(\mathbb{S}^{n-1})} + \|\eta - c\,\mathfrak{M}\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi - \nu\|_{L^2(\mathbb{S}^{n-1})} + \|\nu - c\,\mathfrak{M}\psi\|_{L^2(\mathbb{S}^{n-1})}$$
$$\le C'\varepsilon\big(\|\varphi\|_{L^2(\mathbb{S}^{n-1})} + \|\psi\|_{L^2(\mathbb{S}^{n-1})}\big).$$

On the other hand, for any function $\chi \in L^2(\mathbb{S}^{n-1})$, we have

$$\|\chi - c\,\mathfrak{M}\chi\|_{L^2(\mathbb{S}^{n-1})} \ge \Big(1 - (1+\varepsilon)\max_{m\ge 0}|\mu_m|\Big)\|\chi\|_{L^2(\mathbb{S}^{n-1})},$$

so we can conclude that $\varphi = 0$, $\psi = 0$ if $C'\varepsilon < 1 - (1+\varepsilon)\max_{m\ge 0}|\mu_m|$. Thus, in this case, $\varrho = 1$ and, therefore, K is the unit ball.

6 Example 4: non-uniqueness of convex bodies with prescribed volumes of sections and projections [12]

In this section, we shall construct two essentially different convex bodies K_1 and K_2 in $\mathbb{R}^{2n}$ that have equal $(2n-1)$-dimensional volumes of central sections, maximal sections, and projections in every direction. This answers negatively an old question of Bonnesen and Klee in even dimensions. To the best of our knowledge, the case of odd dimensions still remains open. Our exposition will follow [12].

For a convex body K containing the origin in its interior and $e \in \mathbb{S}^{n-1}$, put

$$A_K(e) = \mathrm{vol}_{2n-1}(K \cap e^{\perp}) \quad \text{(the central section volume)},$$
$$M_K(e) = \max_{t\in\mathbb{R}} \mathrm{vol}_{2n-1}(K \cap (e^{\perp} + te)) \quad \text{(the maximal section volume)},$$
$$P_K(e) = \mathrm{vol}_{2n-1}(K|e^{\perp}) \quad \text{(the projection volume)}.$$

We shall construct two bodies of revolution K_1 and K_2 close to the unit ball that cannot be obtained from each other by a rigid motion but satisfy

$$A_{K_1} \equiv A_{K_2}, \quad M_{K_1} \equiv M_{K_2}, \quad \text{and} \quad P_{K_1} \equiv P_{K_2}.$$

The idea is easiest to demonstrate in $\mathbb{R}^2$. We start with the unit disc, which in the usual Cartesian coordinates x_1, x_2 is given by the inequalities $-1 \le x_1 \le 1$, $|x_2| \le f_o(x_1) = \sqrt{1 - x_1^2}$. Now choose a very small $\delta > 0$ and choose two small in C^2 not identically zero functions φ and ψ supported on $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ and $[1 - 2\delta, 1 - \delta]$, respectively.

Put

$$f_1(x_1) = f_o(x_1) + \varphi(x_1) - \varphi(-x_1) + \psi(x_1)$$

and

$$f_2(x_1) = f_o(x_1) - \varphi(x_1) + \varphi(-x_1) + \psi(x_1),$$

and put

$$K_j = \{(x_1, x_2) : -1 \leq x_1 \leq 1,\ |x_2| \leq f_j(x_1)\}, \quad j = 1, 2.$$

If φ and ψ are small enough, K_1 and K_2 are still convex. Also, if the direction of the vector e makes with the x_1-axis an angle not close to $\frac{\pi}{6}$ or $\frac{\pi}{2}$, then $A_{K_j}(e) = M_{K_j}(e) = P_{K_j}(e) = 2$, $j = 1, 2$.

When the angle is close to $\frac{\pi}{6}$, A_{K_j}, M_{K_j}, and P_{K_j} do not feel the perturbation of f_o by ψ in any way, so they are the same as for the convex bodies $\widetilde{K}_1$ and $\widetilde{K}_2$ corresponding to $\widetilde{f}_1(x_1) = f_o(x_1) + \varphi(x_1) - \varphi(-x_1)$ and $\widetilde{f}_2(x_1) = f_o(x_1) - \varphi(x_1) + \varphi(-x_1)$, but $\widetilde{K}_1$ and $\widetilde{K}_2$ are origin-symmetric images of each other, so we have $A_{K_1} = A_{K_2}$, $M_{K_1} = M_{K_2}$, $P_{K_1} = P_{K_2}$ again. Finally, when the angle is close to $\frac{\pi}{2}$, only ψ matters, so $A_{K_1} = A_{K_2} = A_{\overline{K}}$, $M_{K_1} = M_{K_2} = M_{\overline{K}}$, and $P_{K_1} = P_{K_2} = P_{\overline{K}}$, where $\overline{K}$ corresponds to $\overline{f} = f_o + \psi$.

When $n > 1$ (i. e., we are in $\mathbb{R}^4$ and higher) one can instead consider the bodies of revolution

$$K_j = \{(x_1, x_2, \ldots, x_{2n}) : -1 \leq x_1 \leq 1,\ x_2^2 + \cdots + x_{2n}^2 \leq f_j^2(x_1)\},$$

$j = 1, 2$. The above argument holds without change except for the last part when the angle $\theta(e)$ between e and the x_1-axis is close to $\frac{\pi}{2}$ because now it is no longer true that the volumes of the corresponding sections and projections are not influenced by φ.

The projections present no problem. All one has to note is that when the angle between e and the x_1-axis is close to $\frac{\pi}{2}$, the intersection $(K|e^\perp) \cap \{x : |x_1| \leq \frac{3}{4}\}$ is influenced by φ alone and the intersection $(K|e^\perp) \cap \{x : |x_1| \geq \frac{3}{4}\}$ is influenced by ψ alone.

To take care of the sections, note that for any body of revolution

$$K = \{(x_1, \ldots, x_{2n}) : -1 \leq x_1 \leq 1,\ x_2^2 + \cdots + x_{2n}^2 \leq f^2(x_1)\},$$

the volume of the section $K \cap (e^\perp + te)$ depends only on the angle $\theta(e)$ and on $t \in \mathbb{R}$ and can be computed as

$$c_n \sqrt{1 + s^2} \int_{x_-}^{x_+} [f^2(x_1) - (sx_1 + h)^2]^{n-1} dx_1,$$

where $s = \cot\theta(e)$, $h = \frac{t}{\sin\theta(e)}$, and $x_- < x_+$ are the x_1-coordinates of the intersections of the line $y = sx_1 + h$ with the curves $y = \pm f(x_1)$. Since both K_1 and K_2 are close to the unit ball, we always have $t \approx 0$ for both the maximal and the central section of each of them. When $\theta(e) \approx \frac{\pi}{2}$, this implies that for every choice of $\theta(e) \approx \frac{\pi}{2}$ and $t \approx 0$, the points x_- and x_+ are determined by ψ only, so they are the same for K_1 and K_2. Moreover, these points are close to -1 and 1, respectively. If we now make a choice of φ such that

$$\int_{-\frac{3}{4}}^{\frac{3}{4}} [f_1^2(x_1) - (sx_1 + h)^2]^{n-1} dx_1 = \int_{-\frac{3}{4}}^{\frac{3}{4}} [f_2^2(x_1) - (sx_1 + h)^2]^{n-1} dx_1,$$

for all $s, h \in \mathbb{R}$, then, since $f_1 = f_2$ outside $[-\frac{3}{4}, \frac{3}{4}]$, we will have

$$\mathrm{vol}_{2n-1}(K_1 \cap (e^\perp + te)) = \mathrm{vol}_{2n-1}(K_2 \cap (e^\perp + te))$$

as long as $\theta(e)$ is not too far from $\frac{\pi}{2}$ and t is not too far from 0. This is more than enough to conclude that $A_{K_1}(e) = A_{K_2}(e)$ and $M_{K_1}(e) = M_{K_2}(e)$ when $\theta(e) \approx \frac{\pi}{2}$, finishing the construction.

It remains to show that such choice is possible. Expanding the expressions in powers of s and h, we see that it would suffice to ensure that

$$\Gamma_{l,k}(\varphi) = \int_{-\frac{3}{4}}^{\frac{3}{4}} f_1^l(x_1) x_1^k dx_1 - \int_{-\frac{3}{4}}^{\frac{3}{4}} f_2^l(x_1) x_1^k dx_1 = 0$$

for all $k, l \le 2n - 2$, say. Note that $\varphi \mapsto \Gamma(\varphi) = \{\Gamma_{l,k}(\varphi)\}_{l,k=0}^{2n-2}$ is a continuous mapping from the infinite-dimensional linear space $C_0^2([\frac{1}{2} - \delta, \frac{1}{2} + \delta])$ to $\mathbb{R}^{(2n-1)^2}$ and $\Gamma(-\varphi) = -\Gamma(\varphi)$ (when one changes φ by $-\varphi$, f_1 and f_2 swap places). The Borsuk–Ulam theorem (see [11, p. 23]) implies that for every fixed $\varepsilon > 0$, we can choose φ with $0 < \|\varphi\|_{C^2} < \varepsilon$ such that $\Gamma(\varphi) = 0$, which is exactly what we need.

7 Some open questions

We will finish this survey with five open problems that we would like to see resolved. We by no means pretend that they are “the most important” questions in the area or anything like that. What they reflect is just our personal taste and the general lack of understanding of even the most basic things about convex bodies in dimensions 2 and 3. All these problems are well known and we will accompany each of them with a reference to the earliest known to us source where it was raised. We will intentionally abstain from any comments about our own attempts to solve them.

Problem 1 (Bonnesen [3, p. 51]). Does there exist a convex body $K \subset \mathbb{R}^3$ for which all maximal sections and all projections have the same area (possibly different for sections and projections) but which is not a ball?

Problem 2 (Bonnesen [3, p. 51]). Do there exist two convex bodies $K_1, K_2 \subset \mathbb{R}^3$ such that $M_{K_1} \equiv M_{K_2}$ and $P_{K_1} \equiv P_{K_2}$ but K_1 cannot be obtained from K_2 by a rigid motion?

Problem 3 (Gardner [6, Problem 7.6]). Does there exist an origin-symmetric convex body $K \subset \mathbb{R}^3$ such that all perimeters of central sections of K have the same length but K is not a ball?

Problem 4 (Gardner [6, Problem 7.6]). Let K_1 and K_2 be two origin-symmetric convex bodies in $\mathbb{R}^3$ whose central sections have equal perimeters. Does it follow that $K_1 = K_2$?

Problem 5 ([2]). Do there exist two convex bodies $K_1, K_2 \subset \mathbb{R}^2$ containing the disc of radius 1 in their interiors such that for every $e \in S^1$,

$$\mathrm{length}(K_1 \cap (e^\perp + e)) = \mathrm{length}(K_2 \cap (e^\perp + e))$$

but $K_1 \neq K_2$?

We do not know the answer to any of these questions even in a small neighborhood of the unit ball.

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