ASYMPTOTIC GROWTH OF MULTIPLICITY FUNCTIONS

CĂTĂLIN CIUPERCĂ

Abstract. We consider several multiplicity functions associated with a pair of ideals $J \subseteq I$ in a local noetherian ring R. In particular, given an arbitrary ideal J and an element $x \in R$, we show that for each m the multiplicity $f(n)$ of the R-module $(J+xR)^n/(J+xR)^{n-m}J^m$ is eventually a constant $\varphi_{J,x}(m)$ which is non-zero only in the case when x is not integral over J. We study the asymptotic growth of $\varphi_{J,x}(m)$ and some other multiplicity functions.

1. Introduction

Let R be a commutative local noetherian ring and let J be an ideal of R. An element $x \in R$ is said to be integral over J if x satisfies an equation of integral dependence $x^n + a_1x^{n-1}+\cdots+a_{n-1}x+a_n = 0$ with coefficients $a_i \in J^i$. Equivalently, if we denote $I := J+xR$, the element x is integral over J if and only if there exists $n \geq 1$ such that $JI^{n-1} = I^n$, in which case we say that J is a reduction of I. Roughly speaking, this means that the ideals J and $J + xR$ have the same asymptotic power growth. When the ideals are $\mathfrak{m}$-primary, the classical Hilbert-Samuel multiplicity $e(-)$ is a numerical invariant that can be used to characterize reductions of ideals. More precisely, a well know result of Rees shows that if $J \subseteq I$ are $\mathfrak{m}$-primary ideals in a formally equidimensional local ring, then J is a reduction of I if and only if $e(I) = e(J)$. In the literature there are results in several directions that generalize this numerical characterization of reductions to the situation when the ideals are not necessarily $\mathfrak{m}$-primary, in which case the classical Hilbert-Samuel multiplicity is no longer defined. One such direction was initiated by Amao [1] who showed that if $J \subseteq I$ are ideals in noetherian ring such that the length $\lambda(I/J)$ is finite, then the function $\lambda(I^n/J^n)$ is eventually a polynomial function. If the ring R is local, Rees complemented Amao's result and showed in [4] that the degree of this polynomial function is at most the dimension of the ring. Moreover, if R is a formally equidimensional local ring and $J \subseteq I$ with $\lambda(I/J) < \infty$, Rees proved that

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J is a reduction of I if and only if the degree of the polynomial function given by $\lambda(I^n/J^n)$ for $n \gg 0$ is at most $\dim R - 1$. This extends the original characterization of reductions given by the Hilbert-Samuel multiplicity in the case of $\mathfrak{m}$-primary ideals because $\lambda(I^n/J^n) = \lambda(R/J^n) - \lambda(R/I^n) = [e(J) - e(I)]n^d/d! +$ lower degree terms, where d is the dimension of R. In the same vein, in a more recent paper [2], a consequence of one of the results proved by Herzog, Puthenpurakal, and Verma shows that given $J \subseteq I$ arbitrary ideals, if we replace the length $\lambda(I^n/J^n)$ with the multiplicity of the R-module I^n/J^n, then the function $\mathrm{e}(I^n/J^n)$ is eventually a polynomial function (see also Remark 2.5). Originally motivated by this result, we begin this note by looking at the degree of this polynomial function and its relation with the concept of reduction. We start in Section 1 by observing that the degree of this function is at most $\dim R - t$, where t is a constant equal to $\dim R/(J^n : I^n)$ for $n \gg 0$. We then show in some particular cases that the maximal degree is attained if and only if J is not a reduction of I (see Proposition 2.4 and Corollary 3.11). However, a general result of the type "J is a reduction of I if and only if the degree of the function $\mathrm{e}(I^n/J^n)$ is at most $\dim R - t - 1$" does not hold in general (see Remark 2.8). As noted in Proposition 2.14, a better behavior with respect to attaining a maximal degree is obtained by considering the integral closures of the powers of the ideals and the function $\mathrm{e}(\overline{I^n}/\overline{J^n})$. In Section 3, we modify our approach by considering for each ideal J, each element $x \in R$, and each $m \in \mathbb{N}$, the numerical function $f_m(n) = \mathrm{e}(I^n/I^{n-m}J^m)$, where $I = J + xR$. Generalizing a result of Rees [4, 2.3], we show that for every m the function $f_m(n)$ is eventually a constant, and therefore we can define $\varphi_{J,x}(m) := f_m(n)$ for $n \gg 0$. We study the asymptotic behavior of the function $\varphi_{J,x}(m)$ in a formally equidimensional local ring and, in particular, show that for every $x \in R$ the function $\varphi_{J,x}(m)$ is eventually a polynomial function which is either identically zero (in the case when x is integral over J) or otherwise of degree exactly $\dim R - \dim R/(\overline{J} : I)$, where $\overline{J}$ denotes the integral closure of J (Theorem 3.13). Finally, when $J \subseteq I = J + xR$ are regular ideals in a formally equidimensional local ring, we show that there exists c such that the numerical function $\mathrm{e}(I^{m+c}/I^cJ^m)$ is eventually polynomial of degree at most $\dim R - \dim R/(\overline{J} : I)$, with equality if and only if x is not integral over J (Proposition 3.15).

2. Degrees of relative multiplicity functions

Throughout this paper, if $H(n)$ is a numerical function that is eventually a polynomial function $P(n)$, we refer to the degree of this polynomial as the degree of $H(n)$.

We also recall that if $(R, \mathfrak{m})$ is a local noetherian ring with unique maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module, the multiplicity $\mathrm{e}(M)$ is defined to be the normalized leading coefficient of the Hilbert function associated with the ideal $\mathfrak{m}$ and the module M, i.e.

$$\lambda(M/\mathfrak{m}^n M) = \frac{\mathrm{e}(M)}{d!} n^d + \text{lower degree terms} \quad (\text{for } n \gg 0)$$

where $d = \dim M$. We caution that a slightly different definition is also used in the literature (notably in [3]) where d is the dimension of the ring R (in which case the multiplicity of M is non-zero if and only if $\dim M = \dim R$).

Amao and Rees ([1] and [4]) proved the following: if $J \subseteq I$ are ideals in R such that the length $\lambda(IM/JM)$ is finite, then $\lambda(I^n M/J^n M)$ is eventually a polynomial function of degree at most $\dim R$.

Remark 2.1. Let us observe that in the Amao-Rees theorem it is enough to assume that $\lambda(I^{n_0} M/J^{n_0} M) < \infty$ for some positive n_0. Indeed, if $\lambda(I^{n_0} M/J^{n_0} M) < \infty$ we have $\lambda(I^{n_0} M/JI^{n_0-1} M) < \infty$ and by applying the Amao-Rees theorem for the R-module $I^{n_0-1}M$ we obtain that $h_1(n) := \lambda(I^n M/J^{n-n_0+1} I^{n_0-1} M)$ is eventually a polynomial function in n of degree at most $\dim R$. On the other hand, keeping in mind that $\lambda(JI^{n_0-1}M/J^{n_0}M)$ is also finite, if we consider the finitely generated graded module $\bigoplus_{n \geq n_0} J^{n-n_0+1} I^{n_0-1} M/J^n M$ over the standard graded ring $\bigoplus_{n \geq 0} J^n$, it follows that $h_2(n) := \lambda(J^{n-n_0+1} I^{n_0-1} M/J^n M)$ is eventually polynomial function of degree at most $\dim R$ (see, for example, [2, 4.6]). Then $\lambda(I^n M/J^n M) = h_1(n) + h_2(n)$ is eventually a polynomial function of degree at most $\dim R$.

We now begin the study of the asymptotic behavior of the function $\mathrm{e}(I^n/J^n)$ where $J \subseteq I$ are arbitrary ideals in a local noetherian ring R. First we observe that the dimension of the R-modules I^n/J^n is eventually constant.

Lemma 2.2. *Let R be a noetherian ring and $J \subseteq I$ ideals in R. Then there exists N such that*

$$\sqrt{(J^n : I^n)} = \sqrt{(J^{n+1} : I^{n+1})} \quad \text{for } n \geq N.$$

Proof. We will show that if $x \in (J^n : I^n)$, then $x^2 \in (J^{n+1} : I^{n+1})$. Indeed, if $x \in (J^n : I^n)$, then $x^2 I^n \subseteq xJ^n$, so $x^2 I^{n+1} \subseteq xIJ^n$. But $IJ^n \subseteq I^n J$, and hence $x^2 I^{n+1} \subseteq xI^n J \subseteq J^{n+1}$. Finally, since R is noetherian, there exists N such that the ascending chain of ideals $\sqrt{(J^n : I^n)}$ stabilizes for $n \geq N$. $\square$

Notation 2.3. Let $J \subseteq I$ be ideals in the noetherian ring R and let $K := \sqrt{(J^n : I^n)}$ for $n \gg 0$. We denote $t(J, I) := \dim R/K$.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring and $J \subseteq I$ proper ideals of R. Let $f(n) := \mathrm{e}(I^n/J^n)$ denote the multiplicity of the R-module I^n/J^n and set $t = t(J, I)$. Then, for $n \gg 0$, $f(n)$ is a polynomial function of degree at most* $\dim R - t$. *Moreover, if $J \subseteq I$ is a reduction, then the degree of $f(n)$ is at most* $\dim R - t - 1$.

Proof. Choose N such that $K := \sqrt{(J^n : I^n)} = \sqrt{(J^{n+1} : I^{n+1})}$ for $n \geq N$, so that $\dim(I^n/J^n) = t = \dim R/K$ for $n \geq N$. Using the associativity formula, for $n \geq N$ we have:

$$\mathrm{e}(I^n/J^n) = \sum_{\mathfrak{p} \supseteq K, \dim R/\mathfrak{p} = t} \mathrm{e}(R/\mathfrak{p}) \lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}}). \tag{2.4.1}$$

For every prime ideal $\mathfrak{p}$ that appears in the above sum, the $R_{\mathfrak{p}}$-module $I^N_{\mathfrak{p}}/J^N_{\mathfrak{p}}$ has finite length, and by Remark 2.1 it follows that $\lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}})$ is eventually a polynomial function of degree at most $\dim R_{\mathfrak{p}}$. Then $f(n)$ is eventually a polynomial function of degree at most $\max\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \supseteq K \text{ and } \dim R/\mathfrak{p} = \dim R/K\}$ which is bounded above by $\dim R - \dim R/K$.

In the case when J is a reduction of I, from the same paper of Rees [4, 2.3] we know that the degree of each of the length functions $\lambda(I^n_{\mathfrak{p}}/J^n_{\mathfrak{p}})$ is at most $\dim R_{\mathfrak{p}} - 1$, and hence the degree of $f(n)$ is at most $\dim R - t - 1$. $\square$

Remark 2.5. As noted in the introduction, the fact that $\mathrm{e}(I^n/J^n)$ is eventually a polynomial function is also observed in [2]. (It is a particular case of the more general result [2, 3.2].) Since we are only interested in the numerical function $\mathrm{e}(I^n/J^n)$, we provided above a direct and short argument which is different from the approach taken in [2] and, more importantly, has the benefit of providing a better bound on the degree. Similarly, we prefer the direct and short proof of Lemma 2.2 to show that $\dim(I^n/J^n)$ is constant for $n \gg 0$, which also follows from [2, 3.3].

We next address the question whether the degree of $\mathrm{e}(I^n/J^n)$ is exactly $\dim R - t(J,I)$ when J is not a reduction of I. We observe the following particular case when the answer is positive.

Proposition 2.6. *Let $(R,\mathfrak{m})$ be a formally equidimensional local ring and $J \subseteq I$ proper ideals of R with J integrally closed on the punctured spectrum. If J is not a reduction of I, then the degree of the function $f(n) = \mathrm{e}(I^n/J^n)$ is exactly $\dim R - t(J,I)$.*

Proof. As before, choose N such that $K := \sqrt{(J^n : I^n)} = \sqrt{(J^{n+1} : I^{n+1})}$ for $n \geq N$ and let $\mathfrak{p}$ be any prime containing K with $\dim R/\mathfrak{p} = t(J,I) = t$. If $\mathfrak{p} = \mathfrak{m}$ (i.e. $t = 0$), then I^N/J^N is of finite length, J^N is not a reduction of I^N, and by [4, 2.1] it follows that $f(Nn) = \lambda(I^{Nn}/J^{Nn})$ has degree $\dim R$ in n, so $f(n)$ has degree $\dim R$. If $\mathfrak{p} \neq \mathfrak{m}$, then $J_\mathfrak{p}$ is integrally closed and hence $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ is not a reduction. By the same result [4, 2.1] it follows that the degree of the function $\lambda(I_\mathfrak{p}^n/J_\mathfrak{p}^n)$ is exactly $\dim R_\mathfrak{p} = \dim R - t$ and by (2.4.1) we get that the degree of $f(n)$ is exactly $\dim R - t(J,I)$. □

Remark 2.7. In the above proposition we only need to know that $J_\mathfrak{p}$ is integrally closed for some $\mathfrak{p} \in \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supseteq K, \dim R/\mathfrak{q} = \dim R/K\}$.

Remark 2.8. In the case when $\lambda(I/J)$ is finite and R is formally equidimensional, as proved by Rees, one has that $\deg f(n) \leq \dim R - 1$ if and only if $J \subseteq I$ is a reduction. However, in general, the inequality $\deg f(n) \leq \dim R - t - 1$ does not imply that J is a reduction of I. Indeed, by (2.4.1), $\deg f(n) \leq \dim R - t - 1$ if and only if $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ is a reduction for all $\mathfrak{p} \supseteq K$ with $\dim R/\mathfrak{p} = t$. On the other hand, $J \subseteq I$ is not a reduction if $J_\mathfrak{p} \subseteq I_\mathfrak{p}$ is not a reduction for some $\mathfrak{p} \supseteq (\overline{J} : I)$.

Motivated by the last statement of Remark 2.8, we next consider the asymptotic behavior of $\mathrm{e}(\overline{I^n}/\overline{J^n})$.

Remark 2.9. If $J \subseteq I$ are ideals in an analytically unramified local ring such that $\lambda(I/J) < \infty$, then $\lambda(\overline{I^n}/\overline{J^n})$ is eventually a polynomial function in n of degree at most $\dim R$ (see [2, 4.11]). In fact, using an argument similar to the one used in Remark 2.1, it is enough to assume that $\lambda(I^{n_0}/J^{n_0}) < \infty$ for some positive n_0.

As in Lemma 2.2, we have the following.

Lemma 2.10. *Let R be an analytically unramified local ring and $J \subseteq I$ ideals in R. Then there exists N such that*

$$\sqrt{(\overline{J^n} : \overline{I^n})} = \sqrt{(\overline{J^{n+1}} : \overline{I^{n+1}})} \quad \textit{for } n \geq N.$$

Proof. Since R is analytically unramified, there exists t such that for $n \geq t$ we have $\overline{I^{n+1}} = \overline{I^n}I$ [3, 9.2.1]. Let $x \in (\overline{J^n} : \overline{I^n})$. Then $x^2\overline{I^n} \subseteq x\overline{J^n} \subseteq x\overline{I^{n-1}J}$, which implies that $x^2\overline{I^n}I \subseteq xI\overline{I^{n-1}J} \subseteq \overline{xI^nJ} \subseteq \overline{\overline{J^n}J} = \overline{J^{n+1}}$. So, for $n \geq t$, we have $x^2\overline{I^{n+1}} \subseteq \overline{J^{n+1}}$. In particular, $x \in \sqrt{(\overline{J^{n+1}} : \overline{I^{n+1}})}$. Finally, there exists $N \geq t$ such that the ascending sequence of ideals $\sqrt{(\overline{J^n} : \overline{I^n})}$ stabilizes for $n \geq N$. □

Notation 2.11. Let $J \subseteq I$ be ideals in an analytically unramified local ring R. If $L := \sqrt{(\overline{J^n} : \overline{I^n})}$ for $n \gg 0$, we denote $\bar{t} = \bar{t}(J, I) = \dim R/L$.

Proposition 2.12. *Let $(R, \mathfrak{m})$ be an analytically unramified local ring and $J \subseteq I$ proper ideals of R. Let $g(n) := \mathrm{e}(\overline{I^n}/\overline{J^n})$ denote the multiplicity of the R-module $\overline{I^n}/\overline{J^n}$ and set $\bar{t} = \bar{t}(J, I)$. Then, for $n \gg 0$, $g(n)$ is a polynomial function of degree at most* $\dim R - \bar{t}$.

Proof. The proof is very similar to the one used for Proposition 2.4. Choose N such that $L := \sqrt{(\overline{J^n} : \overline{I^n})} = \sqrt{(\overline{J^{n+1}} : \overline{I^{n+1}})}$ for all $n \geq N$. From the associativity formula, for $n \geq N$ we have:

$$\mathrm{e}(\overline{I^n}/\overline{J^n}) = \sum_{\mathfrak{p} \supseteq L,\, \dim R/\mathfrak{p} = \bar{t}} \mathrm{e}(R/\mathfrak{p})\lambda(\overline{I^n}_{\mathfrak{p}}/\overline{J^n}_{\mathfrak{p}}). \tag{2.12.1}$$

For every prime ideal $\mathfrak{p}$ that appears in the above sum, the $R_{\mathfrak{p}}$-module $\overline{I^N}_{\mathfrak{p}}/\overline{J^N}_{\mathfrak{p}}$ has finite length, and therefore, as noted in Remark 2.9, $\lambda(\overline{I^n}_{\mathfrak{p}}/\overline{J^n}_{\mathfrak{p}})$ is a polynomial function of degree at most $\dim R_{\mathfrak{p}}$. Then $\deg g(n) \leq \max\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \supseteq L, \dim R/\mathfrak{p} = \dim R/L\} \leq \dim R - \bar{t}$. □

Remark 2.13. In the case when $\lambda(I/J)$ is finite and R is an analytically unramified and formally equidimensional local ring, in [2, 4.11] it is also proved that the degree of the function $\lambda(\overline{I^n}/\overline{J^n})$ is exactly $\dim R$ when J is not a reduction of I. This the analogue of Rees's result for the filtration of integral closure of powers of ideals. In contrast to what we noted in Remark 2.8, this result does generalize to the case when I/J is not necessarily of finite length, as shown in the following proposition.

Proposition 2.14. *Let $(R,\mathfrak{m})$ be an analytically unramified, formally equidimensional local ring and let $J \subseteq I$ be proper ideals of R such that J is not a reduction of I. Let $\bar{t} = \bar{t}(J,I)$. Then the degree of $g(n) = \mathrm{e}(\overline{I^n}/\overline{J^n})$ is exactly* $\dim R - \bar{t}$.

Proof. Since R is analytically unramified there exists a positive integer t such that $(\overline{I^t})^k = \overline{I^{tk}}$ for all $k \geq 1$, i.e. $\overline{I^t}$ is a normal ideal. Similarly for J, there exists t' such that $\overline{I^{t'}}$ is a normal ideal. Also, we can choose the same t that works for both I and J, i.e. both $\overline{I^t}$ and $\overline{J^t}$ are normal ideals. Since J is not a reduction of I we know that $\overline{J^t} \neq \overline{I^t}$. By applying Proposition 2.6, we obtain that the function $\mathrm{e}((\overline{I^t})^n/(\overline{J^t})^n)$ is eventually a polynomial function of degree exactly $\dim R - t(\overline{J^t}, \overline{I^t})$. Note that $t(\overline{J^t}, \overline{I^t}) = \bar{t}(J,I)$ and hence $g(nt)$ is a polynomial function of n of degree exactly $\dim R - \bar{t}$. Since $g(n)$ is known to be eventually a polynomial function, it follows that the degree of $g(n)$ is exactly $\dim R - \bar{t}$. □

3. A different direction: other multiplicity functions

In this section we study the asymptotic behavior of several multiplicity functions associated with an ideal J and an element x in a noetherian local ring $(R,\mathfrak{m})$.

We begin by considering the following closure operation of ideals.

Definition 3.1. Let I be a proper ideal in a noetherian ring R. For each ideal J of R we denote $J^{(I)} := \bigcup_{n\geq 1}(JI^n : I^n)$.

Remark 3.2. If I has positive height, then $J^{(I)} \subseteq \overline{J}$. This is an immediate consequence of the determinantal trick [3, 1.1.8]. In particular, if $\operatorname{ht} I > 0$ and J is integrally closed, then $J^{(I)} = J$.

3.3. Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I and J proper ideals of R, and $P_1, \ldots, P_t$ prime ideals of R that do not contain I. Then there exists $x \in I \setminus (P_1 \cup \ldots \cup P_t)$ and $c \in \mathbb{N}$ such that

$$xR \cap I^n J^m = xI^{n-1}J^m \text{ for } n \geq c \text{ and all } m \geq 0. \tag{3.3.1}$$

For a proof, see [5, Lemma 1.2]. In fact, there exists a non-empty Zariski open $U \subseteq I/\mathfrak{m}I$ and $c \in \mathbb{N}$ such that (3.3.1) holds whenever $\bar{x} \in U \subseteq I/\mathfrak{m}I$. In particular, if $\operatorname{depth}_I R > 0$ and $\{P_1, \ldots, P_t\}$ is the set of associated primes of R, there exists $x \in I$ non-zero-divisor and $c \in \mathbb{N}$ such that $(I^n J^m : x) = I^{n-1}J^m$ for $n \geq c$ and $m \geq 0$. Note that we also have $(I^n : x) = I^{n-1}$ for $n \gg 0$.

Lemma 3.4. *Let $J \subseteq I$ be proper ideals in a local noetherian ring R. Assume that* $\operatorname{depth}_I R > 0$. *Then there exists a positive integer c such that for all $n \geq c$ and all $m \geq 0$ we have*

$$(J^m I^n)^{(I)} = J^m I^n.$$

Proof. Replacing the local ring R with the faithfully flat extension $R[X]_{\mathfrak{m}[X]}$ which always has infinite residue field $(R/\mathfrak{m})(X)$, without loss of generality we may assume that the residue field of R is infinite.

As discussed in (3.3), there exists $x \in I$ non-zero-divisor and $c \in \mathbb{N}$ such that

$$(I^n J^m : x) = I^{n-1}J^m \text{ for } n \geq c \text{ and } m \geq 0.$$

Then $(I^n J^m : I) = I^{n-1}J^m$ for $n \geq c$ and $m \geq 0$, which implies that

$$(I^{n+k}J^m : I^k) = (I^{n+k}J^m : I) : I^{k-1} = \ldots = I^n J^m$$

for $n \geq c$, $m \geq 0$ and $k \geq 1$. In particular, $(J^m I^n)^{(I)} = J^m I^n$ for $n \geq c$ and $m \geq 0$. $\square$

Remark 3.5. If $J \subseteq I$ are ideals such that $J^{(I)} = J$, then

$$\sqrt{(J : I)} = \sqrt{(J^2 : I^2)} = \ldots = \sqrt{(J^n : I^n)} = \ldots.$$

We have already seen in the proof of Lemma 2.2 that $\sqrt{(J^n : I^n)} \subseteq \sqrt{(J^{n+1} : I^{n+1})}$ for all n. On the other hand, if $x \in (J^{n+1} : I^{n+1})$, then $xI^{n+1} \subseteq J^{n+1} \subseteq JI^n$, and hence $xI \subseteq (JI^n : I^n) \subseteq J^{(I)} = J$, which implies that $x \in (J : I)$.

Let J be an ideal in a local noetherian ring R and x an element of R. Let $I := J + xR$. Inspired by [4], we now consider the behavior of the numerical functions $f_m(n) = \mathrm{e}(I^n/I^{n-m}J^m)$ for all $m \geq 1$. The next proposition generalizes [4, 2.3] which only applies in the case $\lambda(I/J) < \infty$.

Proposition 3.6. *Let $J \subseteq I$ be proper ideals in the local noetherian ring R. For each $m \geq 1$ consider the function $f_m(n) := \mathrm{e}(I^n/I^{n-m}J^m)$ defined for $n \geq m$. Assume that I/J is cyclic. Then the following are true:*

(a) For each $m \geq 1$ there exists $c = c(m) \in \mathbb{N}$ such that for all $n \geq c$ we have

$$f_m(n) \geq f_m(n+1).$$

(b) If $J^{(I)} = J$, then $f_m(n) \geq f_m(n+1)$ for all $n \geq m$.

(c) Assume that $\operatorname{depth}_I R > 0$. *Then there exists $c \in \mathbb{N}$ such that for all $m \geq 1$ and all $n \geq m + c$ we have*

$$f_m(n) \geq f_m(n+1).$$

Proof. (a) Since $I = J + xR$ for some $x \in I$, we have

$$I^{n+1} = I^{n-m+1}(J + xR)^m = I^{n-m+1}(J^m + xI^{m-1}) = xI^n + I^{n-m+1}J^m.$$

Then the map

$$\theta : I^n/I^{n-m}J^m \to I^{n+1}/I^{n+1-m}J^m$$

defined by multiplication by x is surjective. On the other hand, the ascending chain of ideals $\{(I^{n-m}J^m : I^n)\}_n$ eventually stabilizes, so there exists $c = c(m)$ such that for all $n \geq c$ the two R-modules in the map θ have the same dimension, and therefore $f_m(n) \geq f_m(n+1)$ for $n \geq c$.

(b) If $x \in (I^{n-m}J^m : I^n)$, then $xI \subseteq (JI^{n-1} : I^{n-1}) \subseteq J^{(I)} = J$, so $(I^{n-m}J^m : I^n) \subseteq (J : I)$. Moreover, $(J^n : I^n) \subseteq (I^{n-m}J^m : I^n)$ for all $n \geq m$, and therefore, by Remark

3.5, $\sqrt{(I^{n-m}J^m : I^n)} = \sqrt{(J : I)}$ for all $n \geq m$, which shows that the two R-modules in the above map have the same support for all $n \geq m$, and therefore $f_m(n) \geq f_m(n+1)$ for all $n \geq m$.

(c) As in the previous cases we consider the surjective R-module homomorphism θ. By Lemma 3.4, there exists c such that $(J^m I^c)^{(I)} = (J^m I^c)$ for $m \geq 1$. Then, for $n \geq m+c$, we have $(I^{n-m}J^m : I^n) = (I^{n-m-c}I^cJ^m : I^{n-m-c}) : I^{m+c} \subseteq (I^cJ^m)^{(I)} : I^{m+c} = I^cJ^m : I^{m+c}$, which implies that the two R-modules in the homomorphism θ have the same support for all $n \geq m+c$. Then $f_m(n) \geq f_m(n+1)$ for $n \geq m+c$ and all $m \geq 1$. $\square$

Remark 3.7. If $J \subseteq I$ and I/J cyclic such that $\lambda(I/J) < \infty$, then $f_m(n) \geq f_m(n+1)$ for all $n \geq m$. This follows because both R-modules of the surjective homomorphism θ have finite length. (Note that in this case $f_m(n) = \lambda(I^n/I^{n-m}J^m)$.) This particular case was proved by Rees in [4, 2.3].

As an immediate consequence we have the following.

Corollary 3.8. *Let $J \subseteq I$ be proper ideals in the local noetherian ring R such that I/J is cyclic, i.e. $I = J + xR$ for some $x \in R$. Then the following hold:*

(a) For each $m \geq 1$ the function $f_m(n)$ is constant for $n \gg 0$, and therefore we can define

$$\varphi(m) = \varphi_{J,x}(m) := f_m(n) \text{ for } n \gg 0.$$

(b) If $J^{(I)} = J$, then $\mathrm{e}(I^m/J^m) \geq \varphi(m)$ for all $m \geq 1$.

(c) If $\operatorname{depth}_I R > 0$, then there exists $c_0 \in \mathbb{N}$ such that $\mathrm{e}(I^{m+c}/I^cJ^m) \geq \varphi(m)$ for all $m \geq 1$ and $c \geq c_0$.

Remark 3.9. Note that in the case when $J \subseteq I = J + xR$ is a reduction, the function φ is identically zero.

We next focus on the properties of the function $\varphi(m)$. We use methods that generalize those used by Rees [4, Section 2] in the case when $\lambda(I/J) < \infty$.

At the beginning we work under the assumption $J^{(I)} = J$.

Theorem 3.10. *Let $(R,\mathfrak{m})$ be a noetherian local ring and $J \subseteq I$ proper ideals of R such that I/J is cyclic and J is not a reduction of I. Assume that $J^{(I)} = J$. Then for all m we have*

$$\varphi(m) = \sum_{\mathfrak{p}\in\Gamma} \mathrm{e}(R/\mathfrak{p})\lambda\Big(\frac{R[It]_{\gamma_\mathfrak{p}}}{Q^m R[It]_{\gamma_\mathfrak{p}}}\Big) \tag{3.10.1}$$

where $Q = ((J:I), Jt)R[It]$, $\gamma_\mathfrak{p} = \mathfrak{p}R[It] + Q$ and the sum is taken over all prime ideals in $\Gamma = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid (J:I) \subseteq \mathfrak{p}, \dim(R/\mathfrak{p}) = \dim R/(J:I)\}$.

Proof. With the assumption $J^{(I)} = J$, since $J^{(I)} = (JI^n : I^n)$ for $n \gg 0$, note that $J \subseteq I$ not reduction simply means $J \neq I$. In the Rees ring $\mathcal{R} = R[It]$ the homogeneous ideal Q has the following decomposition:

$$Q = (J:I) \oplus Jt \oplus JIt^2 \oplus JI^2t^3 \oplus \cdots ,$$

and hence

$$R[It]/Q \cong R/(J:I) \oplus I/J \oplus I^2/JI \oplus I^3/JI^2 \oplus \cdots .$$

Since $I = J + xR$ for some $x \in I$, for each n we have $I^n = (J+xR)^n = x^nR + JI^{n-1}$, and therefore each homogeneous component I^n/JI^{n-1} of $R[It]/Q$ is a cyclic R-module generated by the image of x^n. We claim that in fact $R[It]/Q$ is isomorphic to the polynomial ring $(R/(J:I))[X]$. To show this we will prove the following: if $\beta \in R$ and $\beta\overline{x^n} = \overline{0}$ in I^n/JI^{n-1}, then $\beta \in (J:I)$. Indeed, if $\beta x^n \in JI^{n-1}$, then $\beta I^n \subseteq JI^{n-1}$, so $\beta I \subseteq (JI^{n-1} : I^{n-1}) = J^{(I)} = J$, and hence $\beta \in (J:I)$.

We now consider the homogeneous ideal

$$Q^m = ((J:I), Jt)^m R[It] = ((J:I)^m, (J:I)^{m-1}Jt, \ldots, J^mt^m)R[It].$$

Then, for $n \geq m$, the homogeneous component of degree n of Q^m is $[Q^m]_n = I^{n-m}J^mt^n$, hence $[R[It]/Q^m]_n = I^n/I^{n-m}J^m$. We now consider a filtration with graded submodules M_i of the $R[It]$-module $R[It]/Q^m$:

$$R[It]/Q^m = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_k = (0) \tag{3.10.2}$$

where each M_i/M_{i+1} is isomorphic to $(R[It]/\gamma_i)(-t_i)$ for some $\gamma_i \in \operatorname{Ass}(R[It]/Q^m)$. Since γ_i is a graded prime ideal with $\gamma_i \supseteq Q$ and $R[It]/Q \cong R/(J:I)[X]$, it follows that each γ_i is of one of the following two types:

(a) $\gamma = \mathfrak{p}R[It] + Q$ or

(b) $\gamma = (\mathfrak{p} + It)R[It]$

where $\mathfrak{p}$ is a prime ideal of R with $\mathfrak{p} \supseteq (J:I)$.

Then, for $n \geq m$, we have

$$\mathrm{e}(I^n/I^{n-m}J^m) = \mathrm{e}([R[It]/Q^m]_n) = \sum_{i\in\Delta} \mathrm{e}([M_i/M_{i+1}]_n)$$

where the last sum is taken over all i such that the dimension of the factor $[M_i/M_{i+1}]_n$ is equal to $\dim(I^n/I^{n-m}J^m)$ which, as already shown in the proof of Proposition 3.6 (b), is equal to $\dim R/(J:I)$. Also note that in the above sum, if n is sufficiently large, we only need to consider factors M_i/M_{i+1} that are isomorphic to $R[It]/\gamma$ where γ is a prime of type (a). Indeed, if γ is of type (b), then $[(R[It]/\gamma)(-t_i)]_n = 0$ for $n \gg 0$ (in fact, for $n \geq t_i + 1$). Moreover, if γ is of type (a), then $[(R[It]/\gamma)(-t_i)]_n \cong R/\mathfrak{p}$ for $n \gg 0$. Also, if $\gamma = \mathfrak{p}R[It] + Q$ with $\mathfrak{p}$ minimal over $(J:I)$ such that $\dim R/\mathfrak{p} = \dim R/(J:I)$, the number of factors M_i/M_{i+1} in (3.10.2) that are isomorphic to $R[It]/\gamma$ (with various shifting degrees) is given by $\lambda(R[It]_\gamma/Q^m R[It]_\gamma)$. From these observations it follows that for $n \gg 0$ we have

$$\mathrm{e}(I^n/I^{n-m}J^m) = \sum_{\mathfrak{p}\in\Gamma} \mathrm{e}(R/\mathfrak{p})\lambda\Big(\frac{R[It]_{\gamma_\mathfrak{p}}}{Q^m R[It]_{\gamma_\mathfrak{p}}}\Big)$$

where the sum is taken over all prime ideals $\mathfrak{p}$ in $\Gamma = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid (J:I) \subseteq \mathfrak{p}, \dim(R/\mathfrak{p}) = \dim R/(J:I)\}$. Since, by definition, $\varphi(m) = \mathrm{e}(I^n/I^{n-m}J^m)$ for $n \gg 0$, the conclusion of the theorem follows. $\square$

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring and $J \subseteq I$ proper ideals of R with* $\operatorname{ht} I > 0$ *such that I/J is cyclic and J is not a reduction of I. Assume that $J^{(I)} = J$. Then*

(a) *For $m \gg 0$ the function $\varphi(m)$ is a polynomial function of degree* $\dim R - t(J, I)$;

(b) *For* $m \gg 0$ *the function* $f(m) = \mathrm{e}(I^m/J^m)$ *is a polynomial function of degree* $\dim R - t(J,I)$.

Proof. For part (a), note that from (3.10.1) it follows that for $m \gg 0$ the function $\varphi(m)$ is a polynomial function of degree $\max\{\dim R[It]_{\gamma_\mathfrak{p}} \mid \mathfrak{p} \in \Gamma, \gamma_\mathfrak{p} = \mathfrak{p}R[It] + Q\}$. However, for any $\mathfrak{p} \in \Gamma$ we have $R[It]/\gamma_\mathfrak{p} \cong (R/\mathfrak{p})[X]$, hence $\operatorname{ht}\gamma_\mathfrak{p} = \dim R[It] - \dim R[It]/\gamma_\mathfrak{p} = \dim R - \dim R/(J : I)$ and, as noted in Remark 3.5, $t(J,I) = \dim R/(J : I)$.

For part (b), by Proposition 2.4 we already know that for $m \gg 0$ the function $f(m) = \mathrm{e}(I^m/J^m)$ is polynomial of degree at most $\dim R - t(J,I)$. On the other hand, under our assumptions, by Corollary 3.8 (b) we also know that $f(m) \geq \varphi(m)$ for all m, hence $f(m)$ is of degree $\dim R - t(J,I)$. □

Remark 3.12. If $J \subseteq I$ are ideals in a noetherian ring R, then the ascending chain of ideals $(JI^n : I^{n+1})$ stabilizes, hence there exists c such that $K := \bigcup_{n\geq 0}(JI^n : I^{n+1}) = (JI^c : I^{c+1})$. Also note that $\sqrt{K} = \sqrt{(\overline{J} : I)}$. Indeed, if $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \supseteq (\overline{J} : I)$, then $I_\mathfrak{p} \not\subseteq \overline{J}_\mathfrak{p}$, i.e. $J_\mathfrak{p}$ is not a reduction of $I_\mathfrak{p}$. This means that $J_\mathfrak{p} I_\mathfrak{p}^n \neq I_\mathfrak{p}^{n+1}$ for every n, hence $\mathfrak{p} \supseteq K$. Similarly, if $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \supseteq K$, then $J_\mathfrak{p}$ is not a reduction of $I_\mathfrak{p}$, hence $I_p \not\subseteq \overline{J}_\mathfrak{p}$, which implies that $\mathfrak{p} \supseteq (\overline{J} : I)$.

Without the assumption $J^{(I)} = J$ we are also able to conclude the following.

Theorem 3.13. *Let* $(R,\mathfrak{m})$ *be a formally equidimensional local ring and* $J \subseteq I$ *proper ideals of* R *such that* I/J *is cyclic and* $\operatorname{depth}_I R > 0$. *Assume that* J *is not a reduction of* I. *Then there exists* c *such that for all* m *we have*

$$\varphi(m) = \sum_{\mathfrak{p}\in\Delta} \mathrm{e}(R/\mathfrak{p})\lambda\Big(\frac{R[I^{c+1}t]_{\gamma_\mathfrak{p}}}{Q^m R[I^{c+1}t]_{\gamma_\mathfrak{p}}}\Big) \tag{3.13.1}$$

where $Q = ((JI^c : I^{c+1}), JI^c t)R[I^{c+1}t]$, $\gamma_\mathfrak{p} = \mathfrak{p}R[I^{c+1}t] + Q$ *and the sum is taken over all prime ideals in* $\Delta = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid (\overline{J} : I) \subseteq \mathfrak{p}, \dim(R/\mathfrak{p}) = \dim R/(\overline{J} : I)\}$.

In particular, $\varphi(m)$ *is eventually a polynomial function of degree* $\dim R - \dim R/(\overline{J} : I)$.

Proof. By Lemma 3.4, if c is large enough we have $(JI^c)^{(I)} = JI^c$, and by Remark 3.12, we may also assume that $\bigcup_n (JI^n : I^{n+1}) = (JI^c : I^{c+1})$. Let $J' = JI^c$ and $I' = I^{c+1}$. Note that $J' \subseteq I'$ and I'/J' is still cyclic, for if $I = J + xR$, then $I^{c+1} = JI^c + x^{c+1}R$. Moreover, since $(JI^c)^{(I)} = JI^c$ it follows that $J'^{(I')} = \bigcup_{n\geq 0}(J'I'^n : I'^n) = \bigcup_{n\geq 0}(JI^{c+nc+n} : I^{nc+n}) = (JI^c)^{(I)} = JI^c = J'$, so we are in a situation where we can apply Theorem 3.10 for the pair of ideals $J' \subseteq I'$. Then, for $n \gg 0$, we have

$$\mathrm{e}(I^{(c+1)n}/I^{(c+1)n-m}J^m) = \sum_{\mathfrak{p}\in\Delta} \mathrm{e}(R/\mathfrak{p})\lambda\Big(\frac{R[I^{c+1}t]_{\gamma_\mathfrak{p}}}{Q^m R[I^{c+1}t]_{\gamma_\mathfrak{p}}}\Big)$$

where $Q = ((JI^c : I^{c+1}), JI^c t)R[I^{c+1}t]$, $\gamma_\mathfrak{p} = \mathfrak{p}R[I^{c+1}t] + Q$ and the sum is taken over all prime ideals in $\Delta = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid (JI^c : I^{c+1}) \subseteq \mathfrak{p}, \dim(R/\mathfrak{p}) = \dim R/(JI^c : I^{c+1})\}$. Moreover, by the choice of c, from Remark 3.12 we have $\sqrt{(\overline{J} : I)} = \sqrt{(JI^c : I^{c+1})}$. Since $\varphi(m) = \mathrm{e}(I^n/I^{n-m}J^m)$ for $n \gg 0$, the first conclusion of the theorem follows. In addition, by an argument similar to the one used in Corollary 3.11, it follows that for $m \gg 0$ the function $\varphi(m)$ becomes a polynomial of degree $\dim R - \dim R/(JI^c : I^{c+1}) = \dim R - \dim R/(\overline{J} : I)$. □

We next look at the the asymptotic behavior of another multiplicity function associated with an ideal J and an element $x \in R$. Unlike φ, this function is not always identically zero when x is integral over J.

Proposition 3.14. *Let $J \subseteq I$ be proper ideals in a local noetherian ring R. Then, for each $c \geq 0$, the function*

$$h(m) = h_{J,I,c}(m) := \mathrm{e}(I^{m+c}/I^cJ^m)$$

is eventually a polynomial function of degree at most $\dim R - \dim R/(\overline{J} : I)$*. Moreover, if J is a reduction of I, then the degree of $h(m)$ is at most* $\dim R - \dim R/(\overline{J} : I) - 1$*.*

Proof. First we show that for $m \gg 0$ all the R-modules I^m/I^cJ^m have the same dimension. Let $x \in (J^mI^c : I^{m+c})$. Then

$$x^2 I^{m+c+1} \subseteq xJ^mI^{c+1} \subseteq xJI^{m+c} \subseteq JJ^mI^c = J^{m+1}I^c,$$

i.e $x^2 \in (J^{m+1}I^c : I^{m+c+1})$, so we have an ascending chain of ideals $\{\sqrt{(J^mI^c : I^{m+c})}\}_m$, which eventually stabilizes to an ideal L. In addition, note that

$$\sqrt{(J^mI^c : I^{m+c})} \subseteq \sqrt{(JI^{c+m-1} : I^{c+m})} \subseteq \sqrt{(\overline{J} : I)},$$

so $L \subseteq \sqrt{(\overline{J} : I)}$.

From the associativity formula, for $m \gg 0$ we have

$$\mathrm{e}(I^{m+c}/I^c J^m) = \sum_{\mathfrak{p} \supseteq L,\, \dim R/\mathfrak{p} = \dim R/L} \mathrm{e}(R/\mathfrak{p}) \lambda(I_{\mathfrak{p}}^{m+c}/I_{\mathfrak{p}}^c J_{\mathfrak{p}}^m).$$

For each $\mathfrak{p}$ that appears in the sum above, if we consider the $R_{\mathfrak{p}}$-module $I_{\mathfrak{p}}^c$, we know from [4, Section 1] that $\lambda(I_{\mathfrak{p}}^{m+c}/J_{\mathfrak{p}}^m I_{\mathfrak{p}}^c)$ is eventually a polynomial function of dimension at most $\dim R_{\mathfrak{p}}$, and hence $h(m)$ is eventually a polynomial function of degree at most $\dim R - \dim R/L$. Since $L \subseteq \sqrt{(\overline{J} : I)}$, the first conclusion of the proposition follows. In the case when J is a reduction of I, by [4, 2.3] each of the lengths $\lambda(I_{\mathfrak{p}}^{m+c}/J_{\mathfrak{p}}^m I_{\mathfrak{p}}^c)$ is eventually a polynomial function of dimension at most $\dim R_{\mathfrak{p}} - 1$, and hence the degree of $h(m)$ is at most $\dim R - \dim R/(\overline{J} : I) - 1$. □

Proposition 3.15. *Let R be a formally equidimensional local ring and let $J \subseteq I$ be proper ideals of R such that* $\operatorname{depth}_I R > 0$. *Assume that I/J is cyclic and J is not a reduction of I. Then there exists $c \in \mathbb{N}$ such that the degree of $h(m) = \mathrm{e}(I^{m+c}/I^c J^m)$ is exactly* $\dim R - \dim R/(\overline{J} : I)$.

Proof. By Corollary 3.8 (c), there exists $c \in \mathbb{N}$ such that $h(m) \geq \varphi(m)$ for $m \geq 1$. By the previous proposition and Theorem 3.13, it then follows that $h(m)$ is eventually a polynomial function whose degree is exactly $\dim R - \dim R/(\overline{J} : I)$ if and only if J is not a reduction of I. □

Remark 3.16. We conclude by noting that when $\lambda(I/J) < \infty$, the original result of Rees shows that in the above proposition we can always take $c = 0$.

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Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58108-6050, USA

E-mail address: catalin.ciuperca@ndsu.edu