DERIVATIONS AND RATIONAL POWERS OF IDEALS

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ABSTRACT. If A is a commutative noetherian ring and δ is a derivation on A, we study the integral closure, coefficient ideals, and rational powers of the ideals I of A satisfying $\delta(I) \subseteq I$.

1. INTRODUCTION

Let A be a commutative noetherian ring and $\delta : A \to A$ a derivation on A. We say that an ideal I of A is δ-invariant if $\delta(I) \subseteq I$, where $\delta(I)$ is the ideal generated by all the elements $\delta(f)$ for $f \in I$. The study of these ideals was initiated by Seidenberg [11] and we refer the reader to his original paper for some classical results regarding δ-invariant ideals. In particular, if the ring A contains a field of characteristic zero, Seidenberg proved that all the associated prime ideals of I are δ-invariant and I has a primary decomposition with δ-invariant primary components. More recently, a study of Miranda Neto [8] obtains several interesting results about δ-invariant ideals.

While the δ-invariant ideals have been studied extensively since their introduction, the current results in the literature do not address the properties of their integral closures or other related closures. On the other hand, in the context of rings, a well known theorem of Seidenberg [10, Section 3] shows that if D is a derivation on the quotient field of a noetherian domain A containing a field of characteristic zero and $D(A) \subseteq A$, then $D(\overline{A}) \subseteq \overline{A}$, where $\overline{A}$ is the normalization of A.

The rational powers I_α of an ideal I are generalizations of the integral closures $\overline{I^n}$ of the power ideals of I (2.1). If A contains a field of characteristic zero, δ is a derivation on A, and I is an ideal of A, it is the main aim of this paper to study the behavior of the ideals $\delta(I_\alpha)$.

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In Theorem 2.4 we prove that if I is δ-invariant, then I_α is δ-invariant for every rational $\alpha \geq 0$. In particular, the integral closure of a δ-invariant ideal is δ-invariant.

In Corollary 2.5 we prove that the following are equivalent: (a) $\delta(\overline{I}) \subseteq \overline{I}$; (b) $\delta(I_\alpha) \subseteq I_\alpha$ for every rational $\alpha \geq 0$; (c) $\delta(\overline{I^n}) \subseteq \overline{I^n}$ for some integer $n \geq 1$. Further refinements of these equivalences are obtained in Theorem 2.14 and Corollary 2.15. For an arbitrary ideal I, we show that

$$I : \delta(I) \subseteq I^2 : \delta(I^2) \subseteq \ldots \subseteq I^n : \delta(I^n) \subseteq \overline{I} : \delta(\overline{I}) \subseteq I_\alpha : \delta(I_\alpha)$$

for every integer $n \geq 1$ and every rational $\alpha \geq 0$. Moreover,

$$\overline{I} : \delta(\overline{I}) = \overline{I^n} : \delta(\overline{I^n})$$

for every integer $n \geq 1$. In particular, if I^n is δ-invariant for some $n \geq 1$, then $\overline{I}$ is δ-invariant. The main results are obtained by applying Seidenberg's theorem [10, Section 3] to various Rees-like algebras. Some properties can be recovered by using a different approach that involves the use of the Rees valuations of the ideal I (2.6).

We also show that there are examples of monomial ideals I in a polynomial ring $K[X_1, \ldots, X_d]$ with $\delta(\overline{I}) \not\subseteq \overline{\delta(I)}$, where $\delta = \partial_i$ is a partial derivation (Remark 2.8). Moreover, for the ideal $\partial(I)$ generated by all the partial derivatives of the elements of I, the inclusion $\partial(\overline{I}) \subseteq \overline{\partial(I)}$ also fails in general, even for monomial ideals (Remark 2.13).

From a different perspective of looking at integral closure, in Section 3 we consider the coefficient ideals associated with an arbitrary ideal in a noetherian domain A. Using the approach of Corso, Polini, and Vasconcelos in [3], for each $j = 1, \ldots, d = \dim A$, the j-th coefficient ideal of I is defined by $I_{\{j\}} = \{a \in A \mid \operatorname{ht}(\mathcal{R} :_{\mathcal{R}} at) \geq j\}$ where $\mathcal{R} = A[It, t^{-1}]$. Under some mild conditions on the domain A, the coefficient ideals give a filtration between I and its integral closure $\overline{I}$

$$I \subseteq I_{\{d\}} \subseteq I_{\{d-1\}} \subseteq \ldots \subseteq I_{\{1\}} \subseteq \overline{I},$$

where $I_{\{d\}}$ recovers the Ratliff-Rush ideal $\cup_{n\geq 1}(I^{n+1} : I^n)$ of I [9]. The construction of the coefficient ideals is of such a nature that one may view the integral closure $\overline{I}$ as the 0-th coefficient ideal $I_{\{0\}}$.

We look at the behavior of these ideals under derivations and show, in particular, that if I is δ-invariant, then all the coefficient ideals $I_{\{j\}}$ $(1 \leq j \leq d)$ are δ-invariant (Corollary 3.10). This property, as mentioned earlier in this section, is also satisfied by the integral closure $\overline{I}$. More generally, we are able to prove that if I and J are ideals of a universally catenary domain A such that $I + \delta(I) \subseteq J \subseteq \overline{I}$, then $\delta(I_{\{j\}}) \subseteq J_{\{j\}}$ for every $j = 1, \dots, d$.

Throughout this paper, all rings are commutative with identity. For a ring A, a derivation $\delta : A \to A$ is a $\mathbb{Z}$-module homomorphism that satisfies the equality $\delta(ab) = a\delta(b) + \delta(a)b$ for every $a, b \in A$. We denote by $\mathrm{Der}(A)$ the A-module of derivations on A.

2. Derivations and rational powers of ideals

If A is a subring of B, the set of all elements $x \in B$ that satisfy an equation of integral dependence $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with coefficients $a_i \in A$ is a subring of B, the integral closure of A in B.

2.1 (Integral closure and rational powers of ideals)**.** For an ideal I in a noetherian ring A, an element $x \in A$ is said to be integral over I if it satisfies an equation of integral dependence $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with coefficients $a_i \in I^i$. The elements that are integral over I form an ideal $\overline{I}$ of A, which we refer to as the integral closure of I. A different way to think about the integral closure of an ideal I is by using the (extended) Rees algebra $\mathcal{R} = A[It, t^{-1}]$. The integral closure of $\mathcal{R}$ in $A[t, t^{-1}]$ is $\overline{\mathcal{R}} = \oplus_{n\in\mathbb{Z}} \overline{I^n} t^n$, where $\overline{I^n} = A$ for $n \leq 0$.

If $\alpha = \frac{p}{q}$ is a rational number with p and q positive integers, the α-th rational power of an ideal I is defined by $I_\alpha := \{x \in A \mid x^q \in \overline{I^p}\}$. It can be seen that the definition depends only on the quotient p/q. Moreover, I_α is an ideal of A that is always integrally closed, $I_\alpha \subseteq I_\beta$ for $\alpha \geq \beta$, and $I_\alpha I_\beta \subseteq I_{\alpha+\beta}$ for every α, β. When $\alpha = m$ is a positive integer, we have $I_m = \overline{I^m}$, so one may view rational powers as generalizations of the integral closures of the powers of an ideal. Moreover, $(I^n)_{\frac{1}{n}} = \overline{I}$ for every positive integer n. We refer the reader to [7, 10.5] for a more detailed account of the rational powers and their properties.

Remark 2.2. For a positive integer a, denote $\mathcal{R}_a = A[It^a, t^{-1}] = \oplus_{n\in\mathbb{Z}} V_n t^n$ and let $\overline{\mathcal{R}_a}$ be its integral closure in $A[t, t^{-1}]$. First note that $V_n = I^{\lceil \frac{n}{a} \rceil}$ for $n \geq 1$, where $\lceil \ \rceil$ is the upper ceiling function. Then $\overline{\mathcal{R}_a} = \oplus_{n\in\mathbb{Z}} K_n t^n$ is a graded ring and $K_n = I_{n/a}$ for every $n \geq 1$. Indeed, if $f \in K_n$, then $f^a t^{na} \in K_{na} t^{na}$, hence $f^a t^{na}$ satisfies an equation of integral dependence $(f^a t^{na})^s + a_1 (f^a t^{na})^{s-1} + \cdots + a_s = 0$ with $a_i \in \mathcal{R}_a$. By taking the homogeneous part of t-degree nas in this equation, we may actually assume that $a_i \in I^{ni} t^{nai}$ for $i = 1, \ldots, s$. Writing $a_i = b_i t^{nai}$ with $b_i \in I^{ni}$ produces an equation of integral dependence of f^a over I^n, hence $f \in I_{\frac{n}{a}}$. Conversely, if $f \in I_{\frac{n}{a}}$, an equation of integral dependence of f^a over I^n gives an equation $(f^a t^{na})^s + b_1 t^{na} (f^a t^{na})^{s-1} + \cdots + b_s t^{nas} = 0$ with $b_i \in I^{ni}$, which shows that ft^n is integral over $\mathcal{R}_a = A[It^a, t^{-1}]$. Thus $f \in K_n$.

Remark 2.3. The study of the integral closure of an ideal I in a noetherian ring A can often be reduced to the case when A is an integral domain. In general, $x \in \overline{I}$ if and only if $x \in \overline{I(A/\mathfrak{p})}$ for every minimal prime ideal $\mathfrak{p}$ of A ([7, Proposition 1.1.5]). It follows immediately from definition that a similar property holds for any rational power of I, that is, $x \in I_\alpha$ if and only if $x \in (I(A/\mathfrak{p}))_\alpha$ for every minimal prime ideal $\mathfrak{p}$ of A.

The next result is a consequence of Seidenberg's theorem [10, Section 3] applied to the algebra $\mathcal{R}_a = A[It^a, t^{-1}]$.

Theorem 2.4. *Let A be a noetherian ring containing a field of characteristic zero, $\delta \in \mathrm{Der}(A)$, and I a δ-invariant ideal of A. Then I_α is δ-invariant for every non-negative $\alpha \in \mathbb{Q}$. In particular, the integral closure $\overline{I}$ is δ-invariant.*

Proof. We first note that we can assume that A is an integral domain. Indeed, every minimal prime ideal $\mathfrak{p}$ of A is also δ-invariant [11, Theorem 1], so δ induces a derivation $\delta_{\mathfrak{p}} : A/\mathfrak{p} \to A/\mathfrak{p}$. The ideal $I(A/\mathfrak{p})$ is $\delta_{\mathfrak{p}}$-invariant and, if the theorem holds for integral domains, it follows that

$$\delta(I_\alpha)A/\mathfrak{p} \subseteq \delta_{\mathfrak{p}}((I(A/\mathfrak{p}))_\alpha) \subseteq (I(A/\mathfrak{p}))_\alpha$$

for every minimal prime ideal $\mathfrak{p}$. By Remark 2.3 this implies that $\delta(I_\alpha) \subseteq I_\alpha$.

We now assume that A is an integral domain. We start by writing $\alpha = \frac{N}{a}$ with N, a positive integers. Let $\mathcal{R}_a = A[It^a, t^{-1}] = \oplus_{n \in \mathbb{Z}} V_n t^n$ and denote by $\overline{\mathcal{R}_a}$ the integral closure of $\mathcal{R}$ in $A[t, t^{-1}]$. Then $\overline{\mathcal{R}_a} = \oplus_{n \in \mathbb{Z}} K_n t^n$ is a graded subalgebra of $A[t, t^{-1}]$ with $K_n = A$ for $n \leq 0$ and $K_n = I_{\frac{n}{a}}$ for $n \geq 1$.

Applying δ coefficient-wise, the derivation $\delta : A \to A$ extends to a derivation on $A[t]$ and, through the quotient rule, to a derivation on $Q(A)(t)$ which we denote D. Since I is δ-invariant, we have $\delta(I^k) \subseteq I^{k-1}\delta(I) \subseteq I^k$, so all the powers of I are δ-invariant. This means that $D(\mathcal{R}_a) \subseteq \mathcal{R}_a$ and from Seidenberg's theorem [10, Section 3] it follows that D maps the integral closure of $\mathcal{R}_a$ in its quotient field to itself. But $D(A[t,t^{-1}]) \subseteq A[t,t^{-1}]$, so that $D(\overline{\mathcal{R}_a}) \subseteq \overline{\mathcal{R}_a}$. From this we obtain $D(K_n t^n) \subseteq K_n t^n$, or equivalently, $\delta(K_n) \subseteq K_n$, for all n. Since $I_\alpha = I_{\frac{N}{a}} = K_N$, the conclusion follows. ☐

Corollary 2.5. *Let A be a noetherian ring containing a field of characteristic zero. Let I be an ideal of A and $\delta \in \operatorname{Der}(A)$. The following are equivalent:*

(1) $\delta(\overline{I}) \subseteq \overline{I}$;
(2) $\delta(I_\alpha) \subseteq I_\alpha$ *for every rational* $\alpha \geq 0$;
(3) $\delta(\overline{I^n}) \subseteq \overline{I^n}$ *for every integer* $n \geq 1$;
(4) $\delta(\overline{I^n}) \subseteq \overline{I^n}$ *for some integer* $n \geq 1$.

Proof. Since the rational powers of I and $\overline{I}$ are the same, the implication (1)⇒(2) follows from Theorem 2.4 applied to the integral closure $\overline{I}$. We obtain (4)⇒(1) by applying the same theorem for $\alpha = \frac{1}{n}$ to the ideal $\overline{I^n}$ and using the fact that $(\overline{I^n})_{\frac{1}{n}} = (I^n)_{\frac{1}{n}} = \overline{I}$. ☐

2.6 (Derivations and Rees valuation rings)**.** Let A be noetherian ring containing a field of characteristic zero, $\delta \in \operatorname{Der}(A)$, and I a δ-invariant ideal. We will use the Rees valuations of I to show that $\delta(\overline{I^n}) \subseteq \overline{I^n}$. This will also establish the implication (1)⇒(3) from Corollary 2.5.

Following the same type of argument used at the beginning of the proof of Theorem 2.4, we can assume that A is an integral domain. We then extend the derivation δ to the quotient field $Q(A)$ of A and construct the Rees valuations of I as follows. If $I = (a_1, \ldots, a_s)$, for each i let $S_i = A[I/a_i]$ and $\overline{S_i}$ the integral closure of S_i. The set $\operatorname{Rees}(I)$ of the Rees valuations of I consists of all discrete valuation domains $(\overline{S_i})_{\mathfrak{p}}$, where i varies from 1 to s and $\mathfrak{p}$ varies over all the prime ideals of $\overline{S_i}$ minimal over $a_i\overline{S_i}$. We refer the reader to [7, Section 10.2] for a detailed description of this construction. Let $V = \overline{A[I/a]}_{\mathfrak{p}} \in \operatorname{Rees}(I)$, where $a \in I$. We claim that $\delta(V) \subseteq V$. It is enough to prove that $\delta(A[I/a]) \subseteq A[I/a]$. Indeed, by Seidenberg's theorem [10, Section 3], $\delta(\overline{A[I/a]}) \subseteq \overline{A[I/a]}$ and through the quotient rule $\delta(V) \subseteq V$. As $A[I/a] = \cup_{n\geq 0}(I^n/a^n)$, let $f \in I^n$. Then

$$\delta\Big(\frac{f}{a^n}\Big) = \frac{\delta(f)a^n - na^{n-1}\delta(a)f}{a^{2n}}.$$

Since $\delta(f) \in I^{n-1}\delta(I) \subseteq I^n$ and $\delta(a) \in \delta(I) \subseteq I$, it follows that $\delta(f/a^n) \in I^{2n}/a^{2n}$, and therefore $\delta(A[I/a]) \subseteq A[I/a]$.

Since $\delta(I) \subseteq I$ and $\delta(V) \subseteq V$, it follows that $\delta(IV) \subseteq IV$, for if $x = \sum_i a_i v_i$ with $v_i \in V$, then $\delta(x) = \sum_i \delta(a_i)v_i + \sum_i a_i\delta(v_i) \in IV$. Moreover, for every $n \geq 1$, $\delta(I^nV) \subseteq I^{n-1}V\delta(IV) \subseteq I^nV$.

Finally, from [7, Theorem 10.2.2], for every $n \geq 1$ we obtain

$$\delta(\overline{I^n}) \subseteq \bigcap_{V\in \mathrm{Rees}(I)} \delta(I^nV) \cap A \subseteq \bigcap_{V\in \mathrm{Rees}(I)} I^nV \cap A = \overline{I^n}.$$

For computing concrete examples in polynomial rings, we will often use the following immediate observation.

Remark 2.7. Let $A = K[X_1, \dots, X_d]$ be a polynomial ring over a field K of characteristic zero and $\delta = \frac{\partial}{\partial X_j}$ $(1 \leq j \leq d)$. Let I be a monomial ideal with $I = (X_j^{n_1}m_1, \dots, X_j^{n_k}m_k, m_{k+1}, \dots, m_s)$ where $n_1, \dots, n_k \geq 1$ and the monomials m_i $(1 \leq i \leq s)$ are not divisible by X_j. Then

$$\delta(I) = (X_j^{n_1-1}m_1, \dots, X_j^{n_k-1}m_k, m_{k+1}, \dots, m_s).$$

Indeed, $X_j^{n_i-1}m_i = \frac{1}{n_i}\delta(X_j^{n_i}m_i)$ for $1 \leq i \leq k$ and $m_i = \delta(X_j m_i)$ for $k+1 \leq i \leq s$, and the inclusion $\supseteq$ follows. The other inclusion can also be checked immediately by considering $\delta(m)$ for arbitrary monomials $m \in I$.

Remark 2.8. We have seen already in Theorem 2.4 that if $\delta(I) \subseteq I$, then $\delta(\overline{I}) \subseteq \overline{I}$. It is worth mentioning that a more general result of the type "if $\delta(I) \subseteq J$, then $\delta(\overline{I}) \subseteq \overline{J}$" does not hold. In particular, it is not always true that $\delta(\overline{I}) \subseteq \overline{\delta(I)}$, as one can see in the following example. Let K be a field of characteristic zero, $A = K[X, Y]$ with the derivation $\delta = \frac{\partial}{\partial X} : A \to A$, and $I = (X^3, X^2Y, Y^4)$. Note that $\overline{I} = (X^3, X^2Y, XY^3, Y^4)$. From Remark 2.7 it follows that $\delta(I) = (X^2, XY, Y^4)$ and $\delta(\overline{I}) = (X^2, XY, Y^3)$. The ideal $\delta(I) = (X^2, XY, Y^4)$ is integrally closed and we can see that $Y^3 \in \delta(\overline{I}) \setminus \overline{\delta(I)}$.

Remark 2.9. In general, $\overline{\delta(I)}$ is not contained in $\delta(\overline{I})$. In particular, if I is integrally closed, then $\delta(I)$ is not necessarily integrally closed. Consider for example the ideal $I = (X, Y)$ in $K[X, Y]$ ($\operatorname{char} K = 0$) and the derivation $\delta = X^2 \frac{\partial}{\partial X} + Y^2 \frac{\partial}{\partial Y}$. Then $\delta(I) = (X^2, Y^2)$, which is not integrally closed.

Remark 2.10. The ideal $\overline{I}$ can be δ-invariant even when I is not. For example, let $I = (X^2, Y^2)$ in $K[X, Y]$ ($\operatorname{char} K = 0$) and let $\delta = X(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y})$. Then $\overline{I} = (X^2, XY, Y^2)$ and $\delta(\overline{I}) = X(X, Y)$, so $\overline{I}$ is δ-invariant. On the other hand, we also have $\delta(I) = X(X, Y)$, so I is not δ-invariant.

We now present several results regarding derivations of integral closures and rational powers of arbitrary ideals. For polynomial rings and partial derivatives, the following result was proved in [2, Theorem 4.3]. Essentially the same proof works in the case stated below. For convenience, we present a brief outline of it.

Theorem 2.11. *Let A be a noetherian ring containing a field of characteristic zero. Let I be an ideal of A and $\delta \in \operatorname{Der}(A)$. Then, for every $\alpha \in \mathbb{Q}$ with $\alpha \geq 1$, we have*

$$\delta(I_\alpha) \subseteq I_{\alpha-1}.$$

Proof. With the same type of argument used at the beginning of the proof of Theorem 2.4, we can assume that A is an integral domain.

Write $\alpha = N/a$ with a, N positive integers and let $\mathcal{R}_a = A[It^a, t^{-1}] = \oplus_{n\in\mathbb{Z}} V_n t^n$. Let $\overline{\mathcal{R}_a} = \oplus_{n\in\mathbb{Z}} K_n t^n$ denote the integral closure of $\mathcal{R}_a$ in $A[t, t^{-1}]$. After extending the derivation δ to $Q(A)(t)$, consider the derivation

$$D = \frac{1}{t^a}\delta : Q(A)(t) \to Q(A)(t).$$

As already noted in Remark 2.2, $V_n = I^{\lceil \frac{n}{a} \rceil}$, so that $\delta(V_n) \subseteq V_{n-a}$. This means that $D(\mathcal{R}_a) \subseteq \mathcal{R}_a$ and since $D(A[t, t^{-1}]) \subseteq A[t, t^{-1}]$, by Seidenberg's theorem [10, Section 3] we obtain $D(\overline{\mathcal{R}_a}) \subseteq \overline{\mathcal{R}_a}$, or equivalently, $\delta(K_n) \subseteq K_{n-a}$ for every integer n. On the other hand, as explained in Remark 2.2, we have $K_N = I_{N/a} = I_\alpha$ and $K_{N-a} = I_{(N-a)/a} = I_{\alpha-1}$, so $\partial(I_\alpha) \subseteq I_{\alpha-1}$. □

Example 2.12. For a derivation δ and an ideal I, we know that $\delta(I^n) \subseteq I^{n-1}\delta(I) \subseteq I^{n-1}$ and Theorem 2.11 shows that $\delta(\overline{I^n}) \subseteq \overline{I^{n-1}}$ for every $n \geq 1$. It is perhaps worth noting that the inclusion $\delta(\overline{I^n}) \subseteq \overline{I^{n-1}}\delta(\overline{I})$ does not hold in general. We present an example where $\delta(\overline{I^2}) \not\subseteq \overline{I}\delta(\overline{I})$. We need to start with an ideal I such that $\overline{I^2} \neq (\overline{I})^2$. For this, we take $I = (X^2, Y^3, Z^7) \subseteq K[X, Y, Z]$ where K is a field of characteristic zero, an ideal that appears in [7, Exercise 1.14, Section 1.8], and $\delta = \frac{\partial}{\partial X}$. First note that $(XY^2Z^6)^{20} = (YZ)(X^2)^{10}(Y^3)^{13}(Z^7)^{17} \in I^{40}$. Then $XY^2Z^6 \in \overline{I^2}$, so $Y^2Z^6 \in \delta(\overline{I^2})$. On the other hand, $\overline{I} = (X^2, Y^3, XY^2, XYZ^2, XZ^4, Y^2Z^3, YZ^5, Z^7)$, $\delta(\overline{I}) = (X, Y^2, YZ^2, Z^4)$ and $Y^2Z^6 \notin \overline{I}\delta(\overline{I})$. The computations were also checked with Macaulay2 [4].

Remark 2.13. Regarding Remark 2.8, even if we consider the ideal $\partial(I)$ generated by all the partial derivatives of the elements of an ideal I in a polynomial ring $K[X_1, \dots, X_d]$, it is still not necessarily true that $\partial(\overline{I}) \subseteq \overline{\partial(I)}$. This fails even for monomial ideals. Consider again the ideal $I = (X^2, Y^3, Z^7) \subseteq K[X, Y, Z]$ from Example 2.12, where K is a field of characteristic zero. Since $(XYZ^2)^6 = Z^5(X^2)^3(Y^3)^2Z^7 \in I^6$, we have $XYZ^2 \in \overline{I}$. Thus $YZ^2 = \frac{\partial}{\partial X}(XYZ^2) \in \partial(\overline{I})$. On the other hand, using Remark 2.7, we obtain $\partial(I) = \partial_1(I) + \partial_2(I) + \partial_3(I) = (X, Y^2, Z^6)$ and therefore $\overline{\partial(I)} = (X, Y^2, Z^6, YZ^3)$. Then $YZ^2 \in \partial(\overline{I}) \setminus \overline{\partial(I)}$.

The following results improve the conclusions obtained in Corollary 2.5.

Theorem 2.14. *Let A be a noetherian ring containing a field of characteristic zero and I an ideal of A. Let $\delta \in \mathrm{Der}(A)$. Then*

(a) $I : \delta(I) \subseteq \overline{I} : \delta(\overline{I})$;
(b) $\overline{I} : \delta(\overline{I}) \subseteq I_\alpha : \delta(I_\alpha)$ *for every rational* $\alpha \geq 0$;
(c) $I^n : \delta(I^n) \subseteq I^{n+1} : \delta(I^{n+1})$ *for every integer* $n \geq 1$.

Proof. (a) Let $f \in I : \delta(I)$ and let $\delta_f : A \to A$ be the derivation defined by $\delta_f(g) = f\delta(g)$. Then $\delta_f(I) \subseteq I$ and from Theorem 2.4 we obtain $\delta_f(\overline{I}) \subseteq \overline{I}$. Thus $f\delta(\overline{I}) \subseteq \overline{I}$.

(b) As above, for $f \in \overline{I} : \delta(\overline{I})$ consider the derivation $\delta_f = f \cdot \delta : A \to A$. Then $\delta_f(\overline{I}) \subseteq \overline{I}$ and by applying Theorem 2.4 for the ideal $\overline{I}$ we obtain $\delta_f(\overline{I}_\alpha) \subseteq \overline{I}_\alpha$. Since the rational powers of I and $\overline{I}$ coincide, this shows that $f\delta(I_\alpha) \subseteq I_\alpha$ for every $f \in \overline{I} : \delta(\overline{I})$.

(c) For $x_1, \dots, x_{n+1} \in I$ denote $\hat{x}_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_{n+1}$. Then we have

$$\begin{aligned}(n+1)\delta(x_1 x_2 \cdots x_{n+1}) &= \sum_{i=1}^{n+1} \delta(x_i \hat{x}_i) \\ &= \sum_{i=1}^{n+1} x_i \delta(\hat{x}_i) + \sum_{i=1}^{n+1} \delta(x_i)\hat{x}_i \\ &= \sum_{i=1}^{n+1} x_i \delta(\hat{x}_i) + \delta(x_1 x_2 \cdots x_{n+1}),\end{aligned}$$

so $n\delta(x_1 x_2 \cdots x_{n+1}) = \sum_{i=1}^{n+1} x_i \delta(\hat{x}_i)$. For $f \in I^n : \delta(I^n)$, we then obtain

$$nf\delta(x_1 x_2 \cdots x_{n+1}) = \sum_{i=1}^{n+1} x_i f \delta(\hat{x}_i) \in \sum_{i=1}^{n+1} x_i I^n \subseteq I^{n+1}.$$

Since this holds for arbitrary $x_1, \dots, x_{n+1} \in I$, it follows that $f\delta(I^{n+1}) \subseteq I^{n+1}$. □

Corollary 2.15. *Let A a noetherian ring containing a field of characteristic zero and I an ideal of A. Let $\delta \in \operatorname{Der}(A)$. Then*

(a) $\overline{I} : \delta(\overline{I}) = \overline{I^n} : \delta(\overline{I^n})$ *for every $n \geq 1$;*

(b) $I^n : \delta(I^n) \subseteq \overline{I} : \delta(\overline{I})$ *for every $n \geq 1$;*

(c) *If I^n is δ-invariant for some $n \geq 1$, then $\overline{I}$ is δ-invariant;*

(d) *If I^m is integrally closed for some $m \geq 1$, then $\overline{I} : \delta(\overline{I}) = I^n : \delta(I^n)$ for every $n \geq m$.*

Proof. (a) The inclusion $\overline{I} : \delta(\overline{I}) \subseteq \overline{I^n} : \delta(\overline{I^n})$ follows directly from Theorem 2.14 (b). If we apply Theorem 2.14 (b) to the ideal I^n for $\alpha = \frac{1}{n}$ we also obtain

$$\overline{I^n} : \delta(\overline{I^n}) \subseteq (I^n)_{\frac{1}{n}} : \delta((I^n)_{\frac{1}{n}}),$$

and since $(I^n)_{\frac{1}{n}} = \overline{I}$, the proof is finished.

(b) From part (a) and Theorem 2.14 (a) we have

$$I^n : \delta(I^n) \subseteq \overline{I^n} : \delta(\overline{I^n}) = \overline{I} : \delta(\overline{I}),$$

which also proves part (c).

(d) From parts (a), (b) and Theorem 2.14 (c) we have

$$I^m : \delta(I^m) \subseteq I^{m+1} : \delta(I^{m+1}) \subseteq \ldots \subseteq \overline{I} : \delta(\overline{I}) = \overline{I^m} : \delta(\overline{I^m}).$$

Since I^m is integrally closed, the conclusion follows. ☐

3. Derivations and coefficient ideals

3.1. We begin by describing a slight modification of a construction detailed in the work of Corso, Polini and Vasconcelos [3, p. 216]. Let R be a noetherian domain of dimension d with quotient field $K = Q(R)$. For each integer $j = 1, \dots, d+1$, define $R^{(j)} = \{x \in K \mid \operatorname{ht}(R :_R x) \geq j\}$ so that we obtain a chain of rings

$$R = R^{(d+1)} \subseteq R^{(d)} \subseteq \dots \subseteq R^{(2)} \subseteq R^{(1)} = K.$$

Also recall that an S_2-ification $S_2(R)$ of R is a birational extension of R that is minimal among the finite birational extensions that satisfy the (S_2) property of Serre as R-modules. If R has an S_2-ification, then it is unique and $S_2(R) = R^{(2)}$ [6, Proposition 2.4]. We also note that the S_2-ification $S_2(R)$ exists for a large class of rings R. It exists, for example, when R is a universally catenary domain such that the extension $R \subseteq \overline{R}$ is finite, where $\overline{R}$ is the integral closure of R in its quotient field [5, 5.11.2].

Theorem 3.2. *Let $R \subseteq S$ be a birational integral extension of noetherian domains with quotient field K. Additionally, if $R \neq S$, assume that R is universally catenary. Let $\delta : K \to K$ be a derivation such that $\delta(R) \subseteq S$. Then $\delta(R^{(j)}) \subseteq S^{(j)}$ for every $j = 1, \dots, d+1$.*

Proof. Let $\alpha \in R^{(j)}$, set $I = (R :_R \alpha)$, and let $x \in I$. Since $x\alpha \in R$ we obtain $\alpha\delta(x) + x\delta(\alpha) \in S$, and therefore $x\alpha\delta(x) + x^2\delta(\alpha) \in S$. But $x\alpha \in R$ and $\delta(x) \in S$, so that $x^2\delta(\alpha) \in S$. If we denote $J := \langle x^2 \mid x \in I\rangle \subseteq R$, this shows that

$$JS\delta(\alpha) \subseteq S \text{ for every } \alpha \in R^{(j)}. \tag{3.2.1}$$

By [7, 4.8.6 and B.5.1], for every prime ideal Q of S we have $\operatorname{ht} Q = \operatorname{ht}(Q \cap R)$. Since $\operatorname{ht} J = \operatorname{ht} I \geq j$, this implies that $\operatorname{ht} Q \geq j$ for every prime $Q \supseteq JS$, so $\operatorname{ht} JS \geq j$. From (3.2.1) it now follows that $\delta(R^{(j)}) \subseteq S^{(j)}$. ☐

Remark 3.3. The assumption that R is universally catenary is only used to conclude that $\operatorname{ht} Q = \operatorname{ht}(Q \cap R)$ for every prime ideal Q of S. For $S = R$ this assumption is not needed.

Remark 3.4. The original construction in [3] considers the chain of rings given by $\{x \in \overline{R} \mid \dim(R/(R :_R x)) \leq d - j\}$ $(1 \leq j \leq d+1)$, where $\overline{R}$ is the integral closure of R in K.

Remark 3.5. If R is a noetherian domain containing a field of characteristic zero and $\delta : K \to K$ is a derivation with $\delta(R) \subseteq R$, by Seidenberg's theorem [10, Section 3] we know that $\delta(\overline{R}) \subseteq \overline{R}$ and therefore $\delta(R^{(j)} \cap \overline{R}) \subseteq R^{(j)} \cap \overline{R}$. Moreover, if R has an S_2-ification, then $S_2(R) = R^{(2)}$ is a finite extension of R, and therefore $R^{(j)} \subseteq \overline{R}$ for $j \geq 2$.

Remark 3.6. For a derivation δ on K, while Seidenberg's theorem shows that $\delta(R) \subseteq R$ implies $\delta(\overline{R}) \subseteq \overline{R}$, a more general result of the type "if $R \subseteq S$ is a birational extension and $\delta(R) \subseteq S$, then $\delta(\overline{R}) \subseteq \overline{S}$", does not hold in general. A counterexample can be constructed from the ideal $I = (X^3, X^2Y, Y^4) \subseteq A = K[X,Y]$ $(\operatorname{char} K = 0)$ considered in Remark 2.8. Let $R = A[It, t^{-1}]$, $\delta : Q(A)(t) \to Q(A)(t)$ the derivation induced by the derivation $\delta = \frac{\partial}{\partial X} : A \to A$, and $S = A[\delta(I)t, t^{-1}]$. We have already seen in Remark 2.8 that $\delta(I) = (X^2, XY, Y^4)$, so $I \subseteq \delta(I)$. This implies that $\delta(I^n) \subseteq I^{n-1}\delta(I) \subseteq \delta(I)^n$ for every $n \geq 1$, so $\delta(R) \subseteq S$. On the other hand, from the same Remark 2.8 we know that $Y^3 \in \delta(\overline{I}) \setminus \overline{\delta(I)}$, so $Y^3 t \in \delta(\overline{R})$ but $Y^3 t \notin \overline{S}$.

3.7 (Coefficient ideals)**.** Let A be a noetherian domain of dimension d and let $\mathcal{R} = A[It, t^{-1}]$ be the extended Rees algebra of I.

As in [3], using the construction 3.1 in the context of extended Rees algebras, for $j = 1, \ldots, d+1$ define

$$I_{\{j\}} = \{a \in A \mid at \in \mathcal{R}^{(j+1)}\},$$

which produces a chain of ideals

$$I = I_{\{d+1\}} \subseteq I_{\{d\}} \subseteq \ldots \subseteq I_{\{1\}}.$$

We refer to $I_{\{j\}}$ as the j-th coefficient ideal of I. We can also define $I_{\{0\}} := \overline{I}$, the integral closure of I. If $\mathcal{R}$ has an S_2-ification $S_2(\mathcal{R})$, we have $S_2(\mathcal{R}) = \mathcal{R}^{(2)} \subseteq \overline{\mathcal{R}}$, hence $I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}$.

The S_2-ification of $\mathcal{R}$ exists, for example, when A is a formally equidimensional analytically unramified local domain. Under these assumptions on the ring A, if I is an ideal primary to the maximal ideal of A, it is proved in [1, Theorem 2.5] that $I_{\{1\}}$ is the largest ideal containing I such that $\mathrm{e}_0(I) = \mathrm{e}_0(I_{\{1\}})$ and $\mathrm{e}_1(I) = \mathrm{e}_1(I_{\{1\}})$, where $\mathrm{e}_0, \ldots, \mathrm{e}_d$ denote the Hilbert coefficients of I. More generally, in [3, Theorem 4.2], it is proved that $I_{\{j\}}$ is the largest ideal containing I such that $\mathrm{e}_i(I) = \mathrm{e}_i(I_{\{j\}})$ for $i = 0, \ldots, j$, and therefore the ideals $I_{\{j\}}$ coincide with the coefficient ideals introduced by Shah in [12]. (Note that [3] uses a different notational convention for the coefficient ideals.)

The following is an immediate consequence of Theorem 3.2.

Theorem 3.8. *Let A be a noetherian domain of dimension d, $\delta \in \mathrm{Der}(A)$, and I and J ideals of A such that $I + \delta(I) \subseteq J \subseteq \overline{I}$. Additionally, if $I \neq J$, assume that A is universally catenary. Then $\delta(I_{\{j\}}) \subseteq J_{\{j\}}$ for every $j = 1, \ldots, d$.*

Proof. First note that $\delta(I^n) \subseteq I^{n-1}\delta(I) \subseteq J^n$ for every $n \geq 1$. Then, for the integral birational extension $A[It, t^{-1}] \subseteq A[Jt, t^{-1}]$ and the extension of the derivation δ to $Q(A)(t)$, we have $\delta(A[It, t^{-1}]) \subseteq A[Jt, t^{-1}]$. For $a \in I_{\{j\}}$ we have $at \in (A[It, t^{-1}])^{(j+1)}$, and hence, by Theorem 3.2, $\delta(a)t \in (A[Jt, t^{-1}])^{(j+1)}$, i.e., $\delta(a) \in J_{\{j\}}$. □

Applying Theorem 3.8 for $J = I + \delta(I)$ we obtain the following immediate consequences.

Corollary 3.9. *Let A be a universally catenary domain of dimension d, I an ideal of A and $\delta \in \mathrm{Der}(A)$. If $\delta(I) \subseteq \overline{I}$, then $\delta(I_{\{j\}}) \subseteq (I + \delta(I))_{\{j\}}$ for every $j = 1, \ldots, d$.*

Corollary 3.10. *Let A be a noetherian domain of dimension d, $\delta \in \mathrm{Der}(A)$, and I a δ-invariant ideal of A. Then $I_{\{j\}}$ is δ-invariant for every $j = 1, \ldots, d$.*

We now recall same elementary properties of the Veronese subring of a graded ring.

3.11 (Veronese subrings)**.** If $S = \oplus_{n\in\mathbb{Z}} S_n$ is a graded ring, for $k \geq 1$ let $S^{[k]} = \oplus_{n\in\mathbb{Z}} S_{kn}$ be the k-th Veronese subring. There is a one-to-one inclusion preserving correspondence between the set $\mathcal{H}(S)$ of homogeneous prime ideals of S and the set $\mathcal{H}(S^{[k]})$ of homogeneous prime ideals of $S^{[k]}$. If $\varphi : \mathcal{H}(S) \to \mathcal{H}(S^{[k]})$ is defined by $\varphi(P) = P \cap S^{[k]}$ and $\psi : \mathcal{H}(S^{[k]}) \to \mathcal{H}(S)$ is defined by $\psi(Q) = \sqrt{QS}$, then φ and ψ are inverses of each other. In particular, if I and J are a homogeneous ideals of S and $S^{[k]}$, respectively, then $\mathrm{ht}\, I = \mathrm{ht}(I \cap S^{[k]})$ and $\mathrm{ht}\, J = \mathrm{ht}(JS)$.

Before proving an analogue of Theorem 2.11 for coefficient ideals we need to establish the following.

Proposition 3.12. *Let A be an noetherian domain of dimension d, I an ideal of A, and $\mathcal{R} = A[It, t^{-1}]$. Then, for $j = 1, \dots, d$ and every $m \geq 1$, we have*

$$(I^m)_{\{j\}} = \{a \in A \mid at^m \in \mathcal{R}^{(j+1)}\}.$$

Proof. We can identify the extended Rees algebra $\mathcal{R}(I^m)$ of I^m with $\mathcal{R}^{[m]} = \oplus_{n\in\mathbb{Z}} I^{mn}t^{mn}$. For $\alpha t^m \in A[t, t^{-1}]$, let $\mathcal{J} = (\mathcal{R} :_{\mathcal{R}} \alpha t^m) \subseteq \mathcal{R}$ and $\mathcal{L} = (\mathcal{R}^{[m]} :_{\mathcal{R}^{[m]}} \alpha t^m) \subseteq \mathcal{R}^{[m]}$. Note that $\mathcal{L} = \mathcal{J} \cap \mathcal{R}^{[m]}$, so by 3.11 we have $\operatorname{ht} \mathcal{L} = \operatorname{ht} \mathcal{J}$. Therefore, for $a \in A$, we have $at^m \in \mathcal{R}^{(j+1)}$ if and only if $at^m \in {\mathcal{R}^{[m]}}^{(j+1)}$, or equivalently, $a \in (I^m)_{\{j\}}$. $\square$

Theorem 3.13. *Let A be a noetherian domain of dimension d, I an ideal of A, and $\delta \in \operatorname{Der}(A)$. Then, for $j \in \{1, \dots, d+1\}$ and $n \geq 2$, we have*

$$\delta((I^n)_{\{j\}}) \subseteq (I^{n-1})_{\{j\}}.$$

Proof. As done before, we extend the derivation δ to a derivation on $Q(A)(t)$. Let $\mathcal{R} = A[It, t^{-1}]$ and consider the derivation $D = t^{-1}\delta : Q(A)(t) \to Q(A)(t)$. Since $D(\mathcal{R}) \subseteq \mathcal{R}$, by Theorem 3.2 we have $D(\mathcal{R}^{(j+1)}) \subseteq \mathcal{R}^{(j+1)}$, which by Proposition 3.12 implies that $\delta((I^n)_{\{j\}}) \subseteq (I^{n-1})_{\{j\}}$. $\square$

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