INTEGRAL CLOSURE AND GENERIC ELEMENTS

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Abstract. Let $(R, \mathfrak{m})$ be a formally equidimensional local ring with $\operatorname{depth} R \geq 2$ and $I = (a_1, \ldots, a_n)$ an $\mathfrak{m}$-primary ideal in R. The main result of this paper shows that if I is integrally closed, then so is its image modulo a generic element, that is, if $T = R[X_1, \ldots, X_n]/(a_1X_1 + \cdots + a_nX_n)$, then $\overline{IT} = \overline{I}T$.

1. Introduction

Let R be a commutative noetherian ring and I an ideal in R. An element $x \in R$ is said to be integral over I if it satisfies an equation $x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = 0$ with $b_i \in I^i$ for all i. The set of all the elements that are integral over I is an ideal $\overline{I}$, the integral closure of I. If $\varphi : R \to S$ is a ring homomorphism, then $\overline{I}S \subseteq \overline{IS}$, a property referred to as *persistence* (see [7, 1.1.3]). The equality does not necessarily hold; however, note that $\overline{I}S = \overline{IS}$ if and only if $\overline{I}S$ is integrally closed.

For a formally equidimensional local ring $(R, \mathfrak{m})$ with $\operatorname{depth} R \geq 2$ and an $\mathfrak{m}$-primary ideal I, we prove that the integral closure of I is preserved under specialization modulo generic elements. In the language of first general grade reductions (2.2) introduced by Hochster [4], this means that the extension of an integrally closed ideal I to a first general grade reduction of (R, I) is integrally closed, too. More precisely, we prove the following theorem.

Theorem 1. Let $(R, \mathfrak{m})$ be a local ring with $\operatorname{depth} R \geq 2$ and $I = (a_1, a_2, \ldots, a_n)$ an $\mathfrak{m}$-primary ideal in R. Let $S = R[X_1, X_2, \ldots, X_n]$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in S$ and $T = S/\alpha S$. The following hold:

(a) If R is formally equidimensional, then $\overline{IT} = \overline{I}T$;
(b) If R is analytically unramified and Cohen-Macaulay, then $\overline{I^mT} = \overline{I^m}T$ for $m \gg 0$.

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This implies the following local version.

Corollary 2. Let $(R, \mathfrak{m})$ be a local ring with $\operatorname{depth} R \geq 2$ and $I = (a_1, a_2, \ldots, a_n)$ an $\mathfrak{m}$-primary ideal in R. Let $U = R[X_1, X_2, \ldots, X_n]_{\mathfrak{m}[X_1,\ldots,X_n]}$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in U$ and $V = U/\alpha U$. The following hold:

(a) If R is formally equidimensional, then $\overline{IV} = \overline{I}V$;

(b) If R is analytically unramified and Cohen-Macaulay, then $\overline{I^mV} = \overline{I^m}V$ for $m \gg 0$.

Under the assumption that R is an equidimensional, universally catenary ring such that $R/\sqrt{0}$ is analytically unramified, part (a) of Theorem 1 also appears in a 2006 preprint of Hong and Ulrich [5, Theorem 2.1]. Both parts of Corollary 2 were also proved by Itoh [10, Theorem 1] for ideals generated by a system of parameters in analytically unramified Cohen-Macaulay local rings of dimension at least two.

One application of these theorems is their use in proofs based on induction. We exemplify this in the final section of this paper by extending some results regarding integrally closed almost complete intersection ideals in regular local rings obtained by the author in [2].

2. Preliminary Results

All the rings considered in this paper are commutative with identity. If the ring R is noetherian and I is an ideal in R, we denote by $\operatorname{grade} I$ the common length of all the maximal regular sequences contained in I. If the ring R is local with maximal ideal $\mathfrak{m}$, then $\operatorname{grade} \mathfrak{m}$ will be denoted $\operatorname{depth} R$. We also say that the local ring R is formally equidimensional if its completion is an equidimensional ring, that is, $\dim \widehat{R}/P = \dim R$ for all the minimal prime ideals $P \in \operatorname{Spec} \widehat{R}$. In the literature, the local rings with this property are also called quasi-unmixed.

In this section we prove several lemmas which will be used in the proof of the main result.

Remark 2.1. With the notation used in Theorem 1 and Corollary 2, both S and U are faithfully flat extensions of R and $\dim U = \dim R$. In particular, if I is an ideal in R, then $\overline{IS} = \overline{I}S$ and $\overline{IU} = \overline{I}U$ [7, 8.4.2(9)].

The next discussion shows that if Theorem 1 is true for *some* set of generators of I, then the theorem holds for *every* set of generators of I.

2.2. (General grade reductions) Let $I = (a_1, \dots, a_n)$ be an ideal in a noetherian ring R with $\operatorname{grade} I > 0$. The element $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ is a non-zero-divisor on $S = R[X_1, \dots, X_n]$ (it follows inductively from [12, (6.13)]), which implies that the grade of $IS/\alpha S$ in $S/\alpha S$ is one less than the grade of I in R. Introduced by Hochster in [4], $(S/\alpha S, IS/\alpha S)$ is called a first general grade reduction of (R, I). While a first general grade reduction does depend on the choice of generators of I, it can be shown that any two first general grade reductions (T_1, IT_1) and (T_2, IT_2) of (R, I) are equivalent in the following sense: there exist indeterminates $Y_1, \dots, Y_r$ over T_1 and $Z_1, \dots, Z_s$ over T_2, and an R-algebra isomorphism $\phi : T_1[Y_1, \dots, Y_r] \xrightarrow{\cong} T_2[Z_1, \dots, Z_s]$. Note that this implies that $\phi(IT_1[Y_1, \dots, Y_r]) = IT_2[Z_1, \dots, Z_s]$, since ϕ is an R-algebra isomorphism. The existence of this R-algebra isomorphism follows from the proof of [4, Proposition 1]; for the convenience of the reader, we repeat the argument here. Assume that (T_1, IT_1) is obtained with respect to the sequence of generators $b_1, \dots, b_m$ of I and (T_2, IT_2) with respect to $c_1, \dots, c_p$. Let (T_3, IT_3) be the first general grade reduction of (R, I) obtained by using $b_1, \dots, b_m, c_1, \dots, c_p$ as generators of I. Since it is enough to show that (T_1, IT_1) is equivalent to (T_3, IT_3) and (T_2, IT_2) is equivalent to (T_3, IT_3), we can assume from the beginning that $m < p$ and $b_k = c_k$ for $k = 1, \dots, m$. By induction, we can also assume that $p = m+1$, in which case we can take $T_1 = R[X_1, \dots, X_m]/(b_1X_1 + \cdots + b_mX_m)$ and $T_2 = R[X_1, \dots, X_m, Z]/(b_1X_1 + \cdots + b_mX_m + c_{m+1}Z)$, where $X_1, \dots, X_m, Z$ are indeterminates over R. Write $c_{m+1} = r_1b_1 + \cdots + r_mb_m$ ($r_i \in R$) and let $X'_k = X_k + r_kZ$ for $k = 1, \dots, m$. Then $X'_1, \dots X'_m, Z$ are algebraically independent over R and we have the following R-algebra isomorphisms

$$T_2 \cong (R[X'_1, \dots, X'_m]/(b_1X'_1 + \cdots + b_mX'_m))[Z] \cong T_1[Z]$$

Now let us observe that if (T_1, IT_1) and (T_2, IT_2) are two first general grade reductions of (R, I) and $m \geq 1$, then $\overline{I^mT_1} = \overline{I^m}T_1$ if and only if $\overline{I^mT_2} = \overline{I^m}T_2$, or equivalently, $\overline{I^m}T_1$ is integrally closed if and only if $\overline{I^m}T_2$ is integrally closed. Indeed, if $\phi : T_1[Y_1, \dots, Y_r] \xrightarrow{\cong} T_2[Z_1, \dots, Z_s]$ is an R-algebra isomorphism, then $\phi(\overline{I^m}T_1[Y_1, \dots, Y_r]) = \overline{I^m}T_2[Z_1, \dots, Z_s]$

and, by Remark 2.1, it follows that $\overline{I^m}T_1$ is integrally closed if and only if $\overline{I^m}T_2$ is integrally closed.

This shows that if Theorem 1 holds for some set of generators of I, then the theorem holds for every set of generators of I. When proving the main result, this observation allows us to choose the set of generators of I with some extra properties enabled by the assumption that I is an $\mathfrak{m}$-primary ideal in a local ring with depth $R \geq 2$.

Lemma 2.3. *Let R be a noetherian ring and $I = (a_1, a_2, \ldots, a_n)$ an ideal in R with* grade $I > 0$. *Let $a \in R$ be a non-zero-divisor such that* $\operatorname{grade}(I + aR) \geq 2$ *and let $\alpha = a_1X_1 + \cdots + a_nX_n \in S = R[X_1, \ldots, X_n]$. Then a, α is a permutable regular sequence on S. In particular, if the elements $a_1, \ldots, a_n$ are non-zero-divisors and* grade $I \geq 2$, *then a_j, α is a permutable regular sequence on S for all j.*

Proof. Both a and α are non-zero-divisors on S (2.2), so it is enough to prove that α is a non-zero-divisor on S/aS. Since $\operatorname{grade}(I + aR) \geq 2$, there exists $c = r_1a_1 + \cdots + r_na_n \in I$ ($r_i \in R$) such that a, c is a regular sequence on R. By applying the R-algebra automorphism of S that maps X_i to $X_i + r_i$ ($i = 1, \ldots, n$), it follows that it is enough to prove that $a, \alpha + c$ is a regular sequence on S.

Let $f \in S$ such that

$$(a_1X_1 + \cdots + a_nX_n + c)f \in aS. \tag{2.3.1}$$

We want to prove that $f \in aS$. Considering a monomial order on S, let $bX_1^{\alpha_1} \ldots X_n^{\alpha_n}$ be the smallest term of f. The coefficient of the smallest term in the left-hand side of (2.3.1) is bc, so $bc \in aR$, and since a, c is a regular sequence on R, we obtain $b \in aR$. Now

$$(f - bX_1^{\alpha_1} \ldots X_n^{\alpha_n})(a_1X_1 + \cdots + a_nX_n + c) \in aS,$$

and repeating the argument with $f - bX_1^{\alpha_1} \ldots X_n^{\alpha_n}$ instead of f we eventually get that all the coefficients of f belong to aR, and hence $f \in aS$. This proves that $\alpha + c$ is a non-zero-divisor on S/aS. $\square$

Keeping the notation introduced in 2.2 we have the following lemma.

Lemma 2.4. *Let R be a noetherian ring and $I = (a_1, a_2, \dots, a_n)$ an ideal in R with $a_1, a_2, \dots, a_n$ non-zero-divisors ($n \geq 2$) and* $\operatorname{grade} I \geq 2$. *Let $\alpha = a_1X_1 + \cdots + a_nX_n \in S = R[X_1, \dots, X_n]$. Then for each i we have the following isomorphisms of R-algebras*

$$T = S/\alpha S \cong R[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]\Big[\frac{a_1}{a_i}X_1 + \cdots + \frac{a_{i-1}}{a_i}X_{i-1} + \frac{a_{i+1}}{a_i}X_{i+1} + \cdots + \frac{a_n}{a_i}X_n\Big]$$

and

$$T_i := T\Big[\frac{a_1}{a_i}, \dots, \frac{a_n}{a_i}\Big] \cong R[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]\Big[\frac{a_1}{a_i}, \dots, \frac{a_n}{a_i}\Big]$$

Proof. We begin by recalling a well known result: if R is a commutative ring and b, c is a regular sequence on R, then the R-algebra homomorphism $\tau : R[X] \to R[c/b]$ with $\tau(X) = -c/b$ induces an R-algebra isomorphism $R[X]/(bX + c) \cong R[c/b]$ (see, for example, [14, (7.1)]).

Since $\alpha S \cap R = (0)$, we may regard R as embedded in $S/\alpha S$. We may also assume that $i = n$. By Lemma 2.3, $a_1X_1 + \cdots + a_{n-1}X_{n-1}, a_n$ is a permutable regular sequence on $R[X_1, \dots, X_{n-1}]$ and hence we obtain the R-algebra isomorphism

$$\phi\colon S/\alpha S \xrightarrow{\cong} R[X_1, \dots, X_{n-1}]\Big[\frac{a_1}{a_n}X_1 + \cdots + \frac{a_{n-1}}{a_n}X_{n-1}\Big]$$

which maps $X_i + \alpha S$ to X_i, for $1 \leq i \leq n-1$, and $X_n + \alpha S$ to $-\frac{a_1}{a_n}X_1 - \cdots - \frac{a_{n-1}}{a_n}X_{n-1}$. Since $\phi(a_i) = a_i$ for all i, this induces the isomorphism

$$(S/\alpha S)\Big[\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}\Big] \cong R[X_1, \dots, X_{n-1}]\Big[\frac{a_1}{a_n}X_1 + \cdots + \frac{a_{n-1}}{a_n}X_{n-1}\Big]\Big[\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}\Big],$$

that is,

$$(S/\alpha S)\Big[\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}\Big] \cong R[X_1, \dots, X_{n-1}]\Big[\frac{a_1}{a_n}, \dots, \frac{a_{n-1}}{a_n}\Big].$$

□

2.5. (Superficial elements) Let R be a noetherian ring and I an ideal in R. An element $x \in I$ is said to be a superficial element of I if there exists $c \in \mathbb{N}$ such that $(I^{n+1} : x) \cap I^c = I^n$ for all $n \geq c$. Such elements exist, for instance, when the ring R has infinite residue fields [7, 8.5.7]. Furthermore, superficial elements of I exist even when we require some extra properties: if R is a local ring with infinite residue field and $K_1, \dots K_m$ are ideals in R not containing I, then there exists a superficial element of I that is not contained in $K_1 \cup \ldots \cup K_m$ [7, 8.5.9].

In particular, if R is a local ring with infinite residue field and I is an ideal in R with $\operatorname{grade} I > 0$, then there exists $x \in I$ a non-zero-divisor superficial element of I. Moreover, there exist elements $a_1, \ldots, a_n$ generating I that are non-zero-divisors and superficial elements. Indeed, we can choose $a_1 \in I$ a non-zero-divisor superficial element of I and then successively take $a_i \in I$ a superficial element outside of $(a_1, \ldots, a_{i-1})$ that avoids all the associated primes of R. Since the ring is noetherian, we eventually have $I = (a_1, \ldots, a_n)$ for some n.

In the same vein, if R is a local ring with infinite residue field and I is an ideal in R with $\operatorname{grade} I \geq 2$, then we can choose $a_1, \ldots, a_n$ generators for I such that a_i, a_j is a regular sequence for all $i \neq j$ and all the elements a_i are superficial. As shown above, start by choosing $a_1 \in I$ a non-zero-divisor superficial element of I and then take $a_i \in I$ a superficial element outside of $(a_1, \ldots, a_{i-1})$ that avoids all the associated prime ideals of the ideals (a_1), $\ldots$, (a_{i-1}). (Since $\operatorname{grade} I \geq 2$, the ideal I is not contained in any of those prime ideals.) Eventually we obtain $I = (a_1, \ldots, a_n)$ for some n. Note that since R is local, a permutation of a regular sequence is a regular sequence, too.

We mention here that a non-zero-divisor $x \in I$ is a superficial element of I if and only if, for all n sufficiently large, $(I^{n+1} : x) = I^n$ [7, 8.5.3]. This also implies that for a non-zero-divisor superficial element x of I we have $(\overline{I^{n+1}} : x) = \overline{I^n}$ for all n. Indeed, let N be such that $(I^{k+1} : x) = I^k$ for $k \geq N$ and let $y \in (\overline{I^{n+1}} : x)$. By [7, 6.8.12], there exists c that avoids all the minimal prime ideals of R and a positive M such that $cy^t x^t \in I^{(n+1)t}$ for all $t \geq M$. If $t \geq \max\{M, N/n\}$, we have $cy^t \in (I^{(n+1)t} : x^t) = I^{nt}$, and by using again the characterization of the integral closure from [7, 6.8.12] we obtain $y \in \overline{I^n}$.

If $I = (a_1, \ldots, a_n)$, by passing to $S = R[X_1, \ldots, X_n]$, the element $\alpha = a_1X_1 + \cdots + a_nX_n \in IS$ is sufficiently general in the above sense. More exactly we have the following proposition.

Proposition 2.6. *Let R be a noetherian ring, $I = (a_1, \ldots, a_n)$ an ideal with* $\operatorname{grade} I > 0$ *and $\alpha = a_1X_1 + \cdots + a_nX_n \in S = R[X_1, \ldots, X_n]$. Then $(\overline{I^m}S : \alpha) = \overline{I^{m-1}}S$ for all m and $(I^mS : \alpha) = I^{m-1}S$ for $m \gg 0$.*

Proof. By localizing at the prime ideals that contain I we may assume that R is a local ring with maximal ideal $\mathfrak{m}$. Moreover, by replacing R with the faithfully flat extension $R(Z) := R[Z]_{\mathfrak{m}R[Z]}$, we may also assume that R has infinite residue field. Now let us observe that we may assume that a_1 is a non-zero-divisor and a superficial element of I. Indeed, since $\operatorname{grade} I > 0$, by a refinement of a prime avoidance argument, it follows that there exist $r_2, \ldots, r_n \in R$ such that $a_1' := a_1 + r_2a_2 + \cdots + r_na_n$ is a non-zero-divisor and a superficial element of I. Note that $I = (a_1', a_2, \ldots, a_n)$. If we consider the R-algebra automorphism $\phi : S \to S$ that maps X_1 to X_1 and X_i to $X_i + r_iX_1$ for $i \geq 2$, we have $\phi(\alpha) = a_1'X_1 + a_2X_2 + \cdots + a_nX_n$, and therefore we may assume that the first generator a_1 of I is a non-zero-divisor superficial element of I. (By repeating the process we can actually make all the generators of I satisfy this property.)

Let $f \in (\overline{I^m}S : \alpha)$. If we consider a monomial order on S with $X_1 < \ldots < X_n$ and $bX_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is the smallest term that appears in f, from $f\alpha \in \overline{I^m}S$ we obtain $ba_1 \in \overline{I^m}$, and hence $b \in (\overline{I^m} : a_1) = \overline{I^{m-1}}$, where the last equality holds because a_1 is a non-zero-divisor superficial element of I (2.5). Replacing f by $f - bX_1^{\alpha_1} \ldots X_n^{\alpha_n}$ and repeating the argument we eventually obtain that all the coefficients of f are in $\overline{I^{m-1}}$. Similarly, if $f \in (I^mS : \alpha)$, all the coefficients of f are in $(I^m : a_1) = I^{m-1}$ for $m \gg 0$. □

Remark 2.7. If I is an ideal generated by a regular sequence $a_1, \ldots, a_n$, then $(I^k : a_i^l) = I^{k-l}$ for all i and all $k \geq l$. This follows, for instance, from the R-algebra isomorphism between the polynomial ring $(R/I)[X_1, \ldots, X_n]$ and the associated graded ring $\mathcal{G} = \oplus_{n \geq 0} I^n/I^{n+1}$ that sends X_j to $a_j + I^2 \in I/I^2$ (see [1, (1.1.8, 1.1.15)]). Moreover, under the same assumptions, $(\overline{I^k} : a_i^l) = \overline{I^{k-l}}$. Indeed, if $ya_i^l \in \overline{I^k}$, then there exists $c \in R$ that avoids all the minimal prime ideals such that $cy^m a_i^{lm} \in I^{km}$ for $m \gg 0$ (cf. [7, 6.8.12]). Then $cy^m \in (I^{km} : a_i^{lm}) = I^{(k-l)m}$ for $m \gg 0$, thus $y \in \overline{I^{k-l}}$.

In the next lemma we use the notation established in Lemma 2.4.

Lemma 2.8. *Let R be a noetherian ring and $I = (a_1, a_2, \ldots, a_n)$ an ideal in R with $a_1, \ldots, a_n$ non-zero-divisors and superficial elements of I and* $\operatorname{grade} I \geq 2$. *Let $\alpha = a_1X_1 + \cdots + a_nX_n \in S = R[X_1, \ldots, X_n], T = S/\alpha S$ and $T_i := T[a_1/a_i, \ldots, a_n/a_i]$. Then X_j, a_i is a permutable regular sequence on T_i for all i, j.*

Proof. Clearly a_i is a non-zero-divisor on T_i, so it is enough to prove that X_j, a_i is a regular sequence on T_i. If $j \neq i$, by the second isomorphism from Lemma 2.4, it follows that X_j is a non-zero-divisor on T_i, and clearly a_i is a non-zero-divisor on

$$T_i/X_jT_i \cong R[X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]\Big[\frac{a_1}{a_i}, \ldots, \frac{a_n}{a_i}\Big].$$

In the case $i = j$, without loss of generality we may assume that $i = j = n$. The second isomorphism from Lemma 2.4 maps X_n to $-\frac{a_1}{a_n}X_1 - \cdots - \frac{a_{n-1}}{a_n}X_{n-1}$, which, as shown in (2.2), is a non-zero-divisor on $R\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big][X_1, \ldots, X_{n-1}]$. To prove that $\frac{a_1}{a_n}X_1 + \cdots + \frac{a_{n-1}}{a_n}X_{n-1}, a_n$ is a regular sequence on $R\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big][X_1, \ldots, X_{n-1}]$ we apply Lemma 2.3 in the ring $R\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big]$ with $a = a_n$ and the ideal $J = \Big(\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big)$. To be able to do this we will show that $a_n, a_1/a_n$ is a regular sequence on $R\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big]$ and hence $\operatorname{grade}(J, a_n) \geq 2$. Since a_1 is superficial element of I and a non-zero-divisor on R, there exists n_0 such that $I^{n_0+k} : a_1 = I^{n_0+k-1}$ for all $k \geq 0$. Assume that $(a_1/a_n)f = a_ng$ with $f, g \in R\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big]$. There exists N_0 such that for $N \geq N_0$ we have $f' = fa_n^N \in I^N \subseteq R$ and $g' = ga_n^N \in I^N \subseteq R$. Then $a_1f' = a_n^2g'$ and hence, for $N \geq \max\{n_0 - 2, N_0\}$, we have $f' \in (I^{N+2} : a_1) = I^{N+1}$. Finally, $f = f'/a_1^N \in IR\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big] = a_nR\Big[\frac{a_1}{a_n}, \ldots, \frac{a_{n-1}}{a_n}\Big]$. □

The following lemma will play a crucial role in the proof of the main result.

Lemma 2.9. *Let R be a noetherian ring and $I = (a_1, a_2, \ldots, a_n)$ an ideal in R with $a_1, \ldots, a_n$ non-zero-divisors and superficial elements of I. Let $S = R[X_1, X_2, \ldots, X_n]$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in S$ and $T = S/\alpha S$. If $\operatorname{grade} I \geq 2$, then X_j is a non-zero-divisor on $T/\overline{I^mT}$ for all j and all $m \geq 1$.*

Proof. We may assume $j = 1$ and let $f \in T$ such that $fX_1 \in \overline{I^mT}$. Then there exists $c \in T$ that avoids all the minimal prime ideals of T such that $cf^kX_1^k \in I^{mk}T$ for $k \gg 0$. Let $T_i = T\Big[\frac{a_1}{a_i}, \ldots, \frac{a_n}{a_i}\Big]$ $(i = 1, \ldots, n)$. Then $cf^kX_1^k \in I^{mk}T_i = a_i^{mk}T_i$ for all i and $k \gg 0$. By Lemma 2.8, X_1, a_i is a permutable regular sequence on T_i, and hence $cf^k \in a_i^{mk}T_i = I^{mk}T_i$ for $k \gg 0$. We note here that for an arbitrary ideal J in T we have $\cap_{i=1}^n JT_i \cap T = \cup_{r=1}^\infty (JI^rT : I^rT)$. Therefore, for $k \gg 0$, we obtain

$$cf^k \in \cap_{i=1}^n I^{mk}T_i \cap T = \cup_{r=1}^\infty (I^{mk+r}T : I^rT) = \widetilde{I^{mk}T},$$

the Ratliff-Rush closure of $I^{mk}T$ [15, 2.3.1]. Since $\widetilde{I^{mk}T} = I^{mk}T$ for $k \gg 0$ [15, 2.3.2], we have $cf^k \in I^{mk}T$ for $k \gg 0$, or equivalently, $f \in \overline{I^m T}$. This finishes the proof that X_1 is a non-zero-divisor on $T/\overline{I^m T}$. □

3. The Proof of the Main Result

With the preliminaries in place we are now prepared to prove the main result. The next proposition, a particular case of Theorem 1 (a), is essentially due to Itoh. It is proved by Itoh in [10] only for ideals generated by regular sequences. However, by using a result also due to Itoh [9, Lemma 3], it can be proved in the generality stated below.

Proposition 3.1. *Let R be a locally formally equidimensional ring and $I = (a_1, a_2, \dots, a_n)$ a parameter ideal in R of height n with* $\operatorname{grade} I \geq 2$. *Let $S = R[X_1, \dots, X_n]$, $\alpha = a_1X_1 + \cdots + a_nX_n \in S$ and $T = S/\alpha S$. Then $\overline{IT} = \overline{I}T$.*

Proof. By localizing at the prime ideals that contain I, we may assume that the ring R is local. Let $A = R[X_1, \dots, X_{n-1}]_{\mathfrak{m}[X_1,\dots,X_{n-1}]}$, where $\mathfrak{m}$ is the maximal ideal of R, $J = IA$, $\beta = a_1X_1 + \cdots + a_{n-1}X_{n-1}$ and $B = A[\beta/a_n]$. Note that by the discussion (2.2) it is enough to prove the proposition for a special set of generators for I. Since $\operatorname{grade} I > 0$, by a refinement of a prime avoidance argument, there exist $r_1, \dots, r_{n-1} \in R$ such that $a'_n := a_n + r_1a_1 + \ldots + r_{n-1}a_{n-1}$ is a non-zero-divisor. Since $I = (a_1, \dots, a_{n-1}, a'_n)$, by replacing a_n with a'_n we may assume that a_n is a non-zero-divisor. Moreover, by Lemma 2.3, β, a_n is a permutable regular sequence on A, and hence, by the result mentioned at the beginning of the proof of Lemma 2.4, we have $B \cong A[X_n]/\alpha A[X_n]$. First we claim that $\overline{J^kB} \cap A = \overline{J^k}$ for every k. Note that J is a parameter ideal in A and a_n, β are part of a minimal set of generators for J. For $w \in \overline{J^kB} \cap A$ we have $wa_n^N \in \overline{J^{k+N}}$ for $N \gg 0$, hence $w \in (\overline{J^{k+N}} : a_n^N) = \overline{J^k}$ (cf. [7, Corollary 6.8.13]).

Next we claim that $\overline{JB} = \overline{J}B$. For $y \in \overline{JB}$, write $y = z/a_n^t$ with $z \in (\beta, a_n)^t \subseteq A$, so that $z \in (\beta, a_n)^t \cap (\overline{J^{t+1}B} \cap A) = (\beta, a_n)^t \cap \overline{J^{t+1}}$. Since R is locally formally equidimensional, by [9, Lemma 3] it follows that $(\beta, a_n)^t \cap \overline{J^{t+1}} = (\beta, a_n)^t\overline{J}$, and therefore $y \in \overline{J}B$.

Now let $f \in S$ with $\bar{f} = f + \alpha S \in \overline{IS/\alpha S}$. If ϕ denotes the isomorphism between $S/\alpha S$ and $R[X_1, \dots, X_{n-1}][\beta/a_n] \subseteq A[\beta/a_n] = B$ we have $\phi(\bar{f}) \in \overline{JB} = \overline{J}B = \overline{I}B$. Then $f \in \overline{I}A[X_n]$ and hence $f \in \overline{I}R[X_1, \dots, X_n]_{\mathfrak{m}[X_1,\dots,X_n]} \cap R[X_1, \dots, X_n] = \overline{I}R[X_1, \dots, X_n]$. □

Remark 3.2. If the ideal I is generated by a regular sequence, then so is J, and from [8, Proposition 6] it follows that the equality $(\beta, a_n)^t \cap \overline{J^{t+1}} = (\beta, a_n)^t \overline{J}$ holds in an arbitrary ring. Therefore in this case the conclusion of Proposition 3.1 is true without assuming that R is locally formally equidimensional.

We now prove the main result of the paper.

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring with* $\operatorname{depth} R \geq 2$ *and* $I = (a_1, a_2, \dots, a_n)$ *an* $\mathfrak{m}$*-primary ideal in* R*. Let* $S = R[X_1, X_2, \dots, X_n]$, $\alpha = a_1X_1 + a_2X_2 + \dots + a_nX_n \in S$ *and* $T = S/\alpha S$*. Then* $\overline{IT} = \overline{I}T$*.*

Proof. By replacing R with the faithfully flat extension $R(Z) := R[Z]_{\mathfrak{m}R[Z]}$ we may assume that R is a local ring with infinite residue field.

We have noted in (2.2) that it is enough to prove the theorem for a special set of generators for I. For a local ring with infinite residue field, one can successively find sufficiently general elements $x_1, \dots, x_d \in I$ that form a minimal reduction of I ($d = \dim R$). Therefore we can choose generators $a_1, \dots, a_n$ for I such that $a_1, \dots, a_d$ is a minimal reduction of I and all the elements $a_1, \dots, a_n$ are non-zero-divisors and superficial elements of I (see discussion in (2.5)).

Let $f \in S$ with $f + \alpha S \in \overline{IT}$. To prove that $f \in \overline{IS} = \overline{I}S$, without loss of generality we may assume that f is a homogeneous polynomial. Since $f + \alpha S \in \overline{IT}$, there exist a positive integer k and polynomials $g_i = g_i(X_1, \dots, X_n) \in I^iS$, $h = h(X_1, \dots, X_n) \in S$ such that

$$f^s + g_1 f^{s-1} + \dots + g_s = (a_1X_1 + \dots + a_nX_n)h. \tag{3.3.1}$$

We will proceed by induction on the degree of f. If the degree of f is zero, i.e., $f \in R$, by evaluating (3.3.1) at $X_1 = \ldots = X_n = 0$ we obtain an equation of integral dependence of f over I, so $f \in \overline{I} \subseteq \overline{IS}$. Now assume that (3.3.1) implies $f \in \overline{IS}$ for any homogeneous polynomial $f \in S$ of degree at most $k-1$ ($k \geq 1$) and let f be a homogeneous polynomial of degree k that satisfies (3.3.1). We will show that $f \in \overline{IS}$ using again induction, but on $e := n-d$. If $n = d$, then I is a parameter ideal and the conclusion follows from Proposition 3.1. Now assume that $e \geq 1$. By evaluating (3.3.1) at $X_n = 0$ we obtain an equality that shows that the coset of $f(X_1, \ldots, X_{n-1}, 0)$ belongs to $\overline{IT'}$ where $T' = R[X_1, \ldots, X_{n-1}]/(a_1X_1 + \cdots + a_{n-1}X_{n-1})$. Since $\overline{IT'} = \overline{(a_1, \ldots, a_{n-1})T'}$ and $f(X_1, \ldots, X_{n-1}, 0)$ is either zero or a homogeneous polynomial of degree k, by the induction hypothesis we have $f(X_1, \ldots, X_{n-1}, 0) \in \overline{IR[X_1, \ldots, X_{n-1}]} \subseteq \overline{IS}$. On the other hand,

$$f(X_1, \ldots, X_n) = f(X_1, X_2, \ldots, X_{n-1}, 0) + X_n f_1(X_1, \ldots, X_n) \tag{3.3.2}$$

for some homogeneous polynomial $f_1 = f_1(X_1, \ldots, X_n) \in S$ of degree $k-1$. Since both $f + \alpha S$ and $f(X_1, X_2, \ldots, X_{n-1}, 0) + \alpha S$ belong to $\overline{IT}$, we have $X_n f_1 + \alpha S \in \overline{IT}$, and by Lemma 2.9 we obtain $f_1 + \alpha S \in \overline{IT}$. Since the degree of f_1 is $k-1$, by the induction hypothesis we have $f_1 \in \overline{IS}$, and (3.3.2) implies that $f \in \overline{IS}$. $\square$

By localizing at the maximal ideal $\mathfrak{m}[X_1, \ldots, X_n]$ of S we also obtain the following local version.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a formally equidimensional local ring with* $\operatorname{depth} R \geq 2$ *and $I = (a_1, a_2, \ldots, a_n)$ an $\mathfrak{m}$-primary ideal in R. Let $U = R[X_1, X_2, \ldots, X_n]_{\mathfrak{m}[X_1, \ldots, X_n]}$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in U$ and $V = U/\alpha U$. Then $\overline{IV} = \overline{I}V$.*

Remark 3.5. Assume that I is an ideal generated by a regular sequence $a_1, \ldots, a_n$. Itoh [8] and Huneke [6] (in the case of rings containing a field) proved that $\overline{I^{m+1}} \cap I^m = \overline{I}I^m$ for all m. Based on this, in a subsequent paper [10], Itoh proved that $\overline{IT} = \overline{I}T$. Let us observe the statement $\overline{IT} = \overline{I}T$ can be used to recover the equality $\overline{I^{m+1}} \cap I^m = \overline{I}I^m$ for all m.

We will use induction on $m + n$. For $m = 0$ there is nothing to prove. For $n = 1$, let $I = (a)$. If $ra^m \in \overline{(a^{m+1})} \cap (a^m)$, then there exists $c \in R$ avoiding all the minimal prime ideals of R such that $cr^s a^{ms} \in (a^{ms+s})$ for $s \gg 0$. Then $cr^s \in (a^s)$ for $s \gg 0$, and hence $r \in \overline{(a)}$. For $n \geq 2$, as in Corollary 3.4, let $V = U/\alpha U$ where $U = R[X_1, X_2, \ldots, X_n]_{\mathfrak{m}[X_1, \ldots, X_n]}$ and $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in U$. Since α is a non-zero-divisor on U (2.2), $\operatorname{grade}(IV) = \operatorname{grade} I - 1 = n - 1$. Also, IV can be generated by $n-1$ elements ($\alpha \notin I\mathfrak{m}U$), and therefore

IV is a complete intersection ideal. Let $y \in \overline{I^{m+1}} \cap I^m$. Then $y + \alpha U \in \overline{I^{m+1}V} \cap I^m V$ and by the induction hypothesis we obtain $y + \alpha U \in I^m\overline{IV}$. Applying Corollary 3.4 we obtain $y \in I^m\overline{I}U + \alpha U$, so that we can write $y = z + u\alpha$ where $z \in I^m\overline{I}U$ and $u \in U$. Then $u\alpha = y - z \in I^m U \cap \overline{I^{m+1}}U$ and using Remark 2.7 we obtain $u \in (I^m U : \alpha) \cap (\overline{I^{m+1}}U : \alpha) = I^{m-1}U \cap \overline{I^m}U = (I^{m-1} \cap \overline{I^m})U$. By the induction hypothesis we then obtain $u \in I^{m-1}\overline{I}U$, so $y \in I^m\overline{I}U \cap R = I^m\overline{I}$.

For an analytically unramified Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension $d \geq 2$ and an ideal I generated by a maximal regular sequence, Itoh [10, Theorem 1(3)] also proved that $\overline{I^m V} = \overline{I^m}V$ for $m \gg 0$. Note that this implies that $\overline{I^m T} = \overline{I^m}T$ for $m \gg 0$. Indeed, all the zero-divisors of the S-module $S/(\overline{I^m}, \alpha)$ are contained in $\mathfrak{m}[X_1, \ldots, X_n]$, and hence $\overline{I^m}V \cap T = \overline{I^m}T$, which implies that $\overline{I^m T} \subseteq \overline{I^m V} \cap T = \overline{I^m}T$.

We will extend this result to arbitrary $\mathfrak{m}$-primary ideals.

Theorem 3.6. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ and $I = (a_1, a_2, \ldots, a_n)$ an $\mathfrak{m}$-primary ideal in R. Let $S = R[X_1, X_2, \ldots, X_n]$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in S$ and $T = S/\alpha S$. Then $\overline{I^m T} = \overline{I^m}T$ for $m \gg 0$.*

Proof. We begin the proof as in Theorem 3.3. We may assume that R has infinite residue field and since it is enough to prove the theorem for some set of generators for I (2.2), choose $a_1, \ldots, a_n$ generators for I such that $(a_1, \ldots, a_d)$ is a minimal reduction of I and all the elements a_i are non-zero-divisors and superficial elements of I (see discussion in (2.5)).

Given an element $f \in S$ with $f + \alpha S \in \overline{I^m T}$, we obtain an equation

$$f^s + g_1 f^{s-1} + \cdots + g_s = (a_1X_1 + \cdots + a_nX_n)h \tag{3.6.1}$$

where $g_i = g_i(X_1, \ldots, X_n) \in I^{mi}S$ and $h = h(X_1, \ldots, X_n) \in S$. We want to prove that for $m \gg 0$ this implies that $f \in \overline{I^m}S + \alpha S$. Note that we may assume that f is homogeneous. We will use induction on the degree of f. If $\deg f = 0$, by setting $X_1 = \ldots = X_n = 0$ in (3.6.1) we obtain $f \in \overline{I^m}$. Now assume that the claim is true for every homogeneous polynomial of degree at most $k-1$ and let f be a homogeneous polynomial of degree k that satisfies (3.6.1). We use again induction on $e := n - d$. If $n = d$, the conclusion follows from the result of Itoh that we mentioned in the discussion preceding this theorem. Assume that

$e \geq 1$. By setting $X_n = 0$ in (3.6.1), the coset of $f(X_1, \ldots, X_{n-1}, 0)$ is an element of $\overline{I^m T'}$ where $T' = R[X_1, \ldots, X_{n-1}]/(a_1X_1 + \cdots + a_{n-1}X_{n-1})$. Since $\overline{I^m T'} = \overline{(a_1, \ldots, a_{n-1})^m T'}$ and $f(X_1, \ldots, X_{n-1}, 0)$ is either zero or a homogeneous polynomial of degree k in $n-1$ variables, by the induction hypothesis we have $f(X_1, \ldots, X_{n-1}, 0) \in \overline{I^m} R[X_1, \ldots, X_{n-1}] + (a_1X_1 + \cdots + a_{n-1}X_{n-1})R[X_1, \ldots, X_{n-1}]$ so that we can write

$$f(X_1, \ldots, X_{n-1}, 0) = p + (a_1X_1 + \cdots + a_{n-1}X_{n-1})q$$

where $p \in \overline{I^m}S$ is a homogeneous polynomial of degree k and $q \in S$ is a homogeneous polynomial of degree $k-1$. On the other hand,

$$f(X_1, \ldots, X_n) = f(X_1, \ldots, X_{n-1}, 0) + X_n f_1(X_1, \ldots, X_n)$$

where $f_1 \in S$ is a homogeneous polynomial of degree $k-1$, and therefore we have

$$f(X_1, \ldots, X_n) = p + (a_1X_1 + \cdots + a_{n-1}X_{n-1} + a_nX_n)q + X_n(f_1 - a_nq). \tag{3.6.2}$$

Since $f + \alpha S \in \overline{I^m T}$ and $p \in \overline{I^m}S$ we have $X_n(f_1 - a_nq) + \alpha S \in \overline{I^m T}$, and by Lemma 2.9 we obtain $(f_1 - a_nq) + \alpha S \in \overline{I^m T}$. Since the polynomial $(f_1 - a_nq)$ is homogeneous of degree at most $k-1$, by the induction hypothesis we have $(f_1 - a_nq) \in \overline{I^m}S + \alpha S$, and by (3.6.2) we obtain $f \in \overline{I^m}S + \alpha S$, which finishes the proof. $\square$

Corollary 3.7. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ and let $I = (a_1, a_2, \ldots, a_n)$ be an $\mathfrak{m}$-primary ideal in R. Denote $U = R[X_1, X_2, \ldots, X_n]_{\mathfrak{m}[X_1,\ldots,X_n]}$, $\alpha = a_1X_1 + a_2X_2 + \cdots + a_nX_n \in U$ and $V = U/\alpha U$. Then $\overline{I^m V} = \overline{I^m}V$ for $m \gg 0$.*

Remark 3.8. Theorems 3.3 and 3.6 and their corollaries can be extended to equimultiple ideals. (In general, the analytic spread $\ell(I)$ of the ideal I is at most the height of I; when the equality holds, we say that I is equimultiple.) More precisely, if R is a locally formally equidimensional ring and I is an equimultiple ideal with $\operatorname{grade} I \geq 2$, then $\overline{IT} = \overline{I}T$ and $\overline{I^m T} = \overline{I^m}T$ for $m \gg 0$. Indeed, if I is equimultiple, then the ideals $\overline{I^m}$ $(m \geq 1)$ have no embedded components ([11]) and, by localizing at the minimal prime ideals of I, the conclusions follow.

4. Applications

As mentioned in the introduction, the results that show that the integral closure is preserved modulo a generic element are useful for proving integral closure statements by induction. As an example of such an application, we extend some results obtained by the author in [2].

4.1. (Almost complete intersection ideals in regular local rings) Let $(A, \mathfrak{m})$ be a regular local ring of dimension d and I an $\mathfrak{m}$-primary almost complete intersection ideal, that is, minimally generated by $d+1$ elements. Under the assumption that A contains a field, the author [2, 3.3, 3.5, 3.7] proved that if I is integrally closed, then I contains $d-2$ regular parameters $x_1, \dots, x_{d-2}$ such that in the 2-dimensional regular local ring $A' = A/(x_1, \dots, x_{d-2})$ the ideal IA' is integrally closed and generated by three elements. The structure of such ideals in 2-dimensional regular local rings is then completely characterized by results of Noh [13]. In addition, we proved that the Rees algebra $\mathcal{R} = \oplus_{n \geq 0} I^n t^n$ is a Cohen-Macaulay normal domain and the associated graded ring $\mathcal{G} = \oplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring with $a(\mathcal{G}) = 1 - d$, where $a(\mathcal{G})$, the a-invariant of G, is defined by $a(\mathcal{G}) = \sup\{i \mid \mathrm{H}_d^M(\mathcal{G})_i \neq 0\}$ with M being the maximal homogeneous ideal of $\mathcal{G}$.

The assumption that A contains a field was needed because the proofs relied upon the following lemma: if A is a regular local ring containing a field, I is an ideal in A with $I \not\subseteq \mathfrak{m}^2$ and $x \in I \setminus \mathfrak{m}^2$, then $\overline{(I/xA)} = \overline{I}/xA$ [2, 3.4]. Using Corollary 3.4, we will prove this lemma without assuming that A contains a field, and hence all the results about integrally closed $\mathfrak{m}$-primary almost complete intersection ideals that we mentioned above hold in an arbitrary regular local ring.

Lemma 4.2. *Let A be a regular local ring, I an ideal in A with $I \not\subseteq \mathfrak{m}^2$ and let $x \in I \setminus \mathfrak{m}^2$. Then $\overline{(I/xA)} = \overline{I}/xA$.*

Proof. It is enough to prove this result for $\mathfrak{m}$-primary ideals I. Indeed, for $f + xA \in \overline{(I/xA)}$ we have $f + xA \in \overline{((I + \mathfrak{m}^k)/xA)}$ for all k, and hence $f \in \overline{I + \mathfrak{m}^k}$ for all k. Since $\cap_k \overline{I + \mathfrak{m}^k} = \overline{I}$ ([7, 6.8.5]), we obtain $f \in \overline{I}$.

Let $I = (a_1, \ldots, a_n)$ and $\alpha = a_1X_1 + \cdots + a_nX_n \in U = A[X_1, \ldots, X_n]_{\mathfrak{m}[X_1,\ldots,X_n]}$. Set $V = U/\alpha U$, denote by M the maximal ideal of V, and let $\phi : A/xA \to V/xV$ be the homomorphism induced by the embedding $A \to V$.

We will use induction on the dimension of A. If $\dim A = 1$, the statement is clear. Assume that $\dim A \geq 2$. Since $\alpha \notin \mathfrak{m}^2U$, V is a regular local ring of dimension one less than the dimension of A. Also, note that $x \in IV \setminus M^2$. By the induction hypothesis, if $y \in A$ with $\bar{y} \in \overline{I/xA}$, then $\phi(\bar{y}) \in \overline{IV/xV} = \overline{IV}/xV$. Furthermore, by Corollary 3.4, we have $\phi(\bar{y}) \in \overline{I}V/xV \cong \overline{I}U/(xU + \alpha U)$, which implies that $y \in \overline{I}U$, and hence $y \in \overline{I}U \cap A = \overline{I}$. $\square$

The following consequence was also noted in [2, 3.5], but now we can drop the assumption that A contains a field. The same proof from [2] will work.

Corollary 4.3. *Let $(A, \mathfrak{m})$ be a d-dimensional regular local ring, and let I be an ideal of A such that the embedding dimension of A/I is at most two. Then $\overline{I^n} = I^{n-1}\overline{I}$ for all $n \geq 1$.*

Remark 4.4. The above corollary also shows that an integrally closed $\mathfrak{m}$-primary complete intersection ideal in a regular local ring is normal. This is a well known result proved by Goto [3, Theorem 3.1] by using different methods.

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