A NUMERICAL CHARACTERIZATION OF THE S_2-IFICATION OF A REES ALGEBRA

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ABSTRACT. Let A be a local ring with maximal ideal $\mathfrak{m}$. For an arbitrary ideal I of A, we define the generalized Hilbert coefficients $j_k(I) \in \mathbb{Z}^{k+1}$ ($0 \leq k \leq \dim A$). When the ideal I is $\mathfrak{m}$-primary, $j_k(I) = (0, \dots, 0, (-1)^k e_k(I))$, where $e_k(I)$ is the classical k^{th} Hilbert coefficient of I. Using these coefficients we give a numerical characterization of the homogeneous components of the S_2-ification of $S = A[It, t^{-1}]$, extending to not necessarily $\mathfrak{m}$-primary ideals the results obtained in [7].

INTRODUCTION

Let $(A, \mathfrak{m})$ be a formally equidimensional local ring and let $I \subseteq J$ be two ideals of A. When I is $\mathfrak{m}$-primary, Rees proved that J is contained in the integral closure $\overline{I}$ of I if and only if I and J have the same multiplicity. Böger [5] extended this result as follows: *let $I \subseteq J \subseteq \sqrt{I}$ be ideals in a formally equidimensional local ring A such that $\ell(I) = \operatorname{ht} I$, where $\ell(I)$ denotes the analytic spread of I. Then I is a reduction of J (equivalently $J \subseteq \overline{I}$) if and only if the $A_{\mathfrak{p}}$-ideals $I_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ have the same multiplicity for every minimal prime divisor $\mathfrak{p}$ of I.*

Using the j-multiplicity defined by Achilles and Manaresi [3] (a generalization of the classical Samuel multiplicity), Flenner and Manaresi [10] gave a numerical characterization of reduction ideals which generalizes Böger's result to arbitrary ideals.

Theorem (Flenner-Manaresi [10]). *Let $I \subseteq J$ be ideals in a formally equidimensional local ring A. Then I is a reduction of J if and only if $j(I_{\mathfrak{p}}) = j(J_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$.*

It is well known that for an integrally closed domain A, the integral closure of the extended Rees algebra $S = A[It, t^{-1}]$ in its quotient field is $\overline{S} = \bigoplus_{n \in \mathbb{Z}} \overline{I^n} t^n$ ($I^n = A$ for $n < 0$), so one could interpret the above results as numerical characterizations of the homogeneous components of $\overline{S}$.

Our motivation comes from the study of the S_2-ification of the same extended Rees algebra $S = A[It, t^{-1}]$. Under some assumptions on the ring A, S has an S_2-ification of the form $\widetilde{S} = \bigoplus_{n \in \mathbb{Z}} I_n t^n$, where $I_n = A$ for $n < 0$. In [7, Theorem 2.4] we proved that if I is primary to the maximal ideal $\mathfrak{m}$, then I_n is the largest ideal

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containing I^n such that $e_i(I_n) = e_i(I^n)$ for $i = 0,1$, where e_0 and e_1 are the first two Hilbert coefficients.

In this paper we use the j-multiplicity of Achilles and Manaresi and a new invariant j_1 to obtain a characterization of $\widetilde{S}$ similar to the one of $\overline{S}$ given by the result of Flenner and Manaresi.

The paper is organized as follows. In the introductory section we establish the notation and recall the main concepts used in the paper.

In the second section we define a generalization of the classical Hilbert coefficients. Achilles and Manaresi [3] defined the so-called j-multiplicity of an ideal I in a local ring A which generalizes to ideals of maximal analytic spread the classical Samuel multiplicity. In a subsequent paper, Achilles and Manaresi [4] also observed that this new invariant can be recovered from the Hilbert polynomial of the bigraded ring $G_{\mathfrak{m}}(G_I(A))$.

This is the point of view we adopt in order to define the coefficients $j_k(I) \in \mathbb{Z}^{k+1}$ ($0 \leq k \leq \dim A$), a generalization of the classical Hilbert coefficients $e_k(I)$. When the ideal I is $\mathfrak{m}$-primary, $j_k(I) = (0,\dots,0,(-1)^k e_k(I))$. We show that these coefficients behave well with respect to general hyperplane sections, one of the main properties one might expect from any generalization of the Hilbert coefficients.

The concept of first coefficient ideals has been introduced by Shah in [19]. He proved that for an $\mathfrak{m}$-primary ideal I in a formally equidimensional ring $(A,\mathfrak{m})$ there exists a unique ideal $I_{\{1\}}$, the first coefficient ideal of I, that is maximal among the ideals containing I for which the first two Hilbert coefficients are equal to those of I. In Section 3 we extend the definition of $I_{\{1\}}$ to not necessarily $\mathfrak{m}$-primary ideals. Our definition is a slight reinterpretation (but necessary for our purpose) of a description of the first coefficient ideals given by Shah.

We then observe that using the new definition of $I_{\{1\}}$ for an arbitrary ideal, we also have $I_n = (I^n)_{\{1\}}$ ($\widetilde{S} = \bigoplus_{n\in\mathbb{Z}} I_n t^n$ is the S_2-ification of the extended Rees algebra S). This follows from the proof of [7, Theorem 2.4] as a direct consequence of an argument due to Heinzer and Lantz [15, 2].

The last section contains the main result of this paper. We give a numerical characterization of the homogeneous components of $\widetilde{S}$ by proving the following theorem.

Theorem. *Let $(A,\mathfrak{m})$ be a formally equidimensional local ring and let $I \subseteq J$ be ideals of positive height. Then the following are equivalent.*

(1) $J \subseteq I_{\{1\}}$.
(2) $j_0(I_{\mathfrak{p}}) = j_0(J_{\mathfrak{p}})$ *and* $j_1(I_{\mathfrak{p}}) = j_1(J_{\mathfrak{p}})$ *for all* $\mathfrak{p} \in \operatorname{Spec}(A)$.

Here $j_0(I) = j(I)$ is the above mentioned j-multiplicity.

In fact, we prove a more general version for modules (but technically simpler for our inductive argument). The proof of the theorem in the 2-dimensional case is a crucial part of the argument (see 4.1, 4.2, and 4.5).

1. Preliminaries

Throughout this paper a local ring $(A,\mathfrak{m})$ will be a commutative Noetherian ring with identity, and unique maximal ideal.

1.1. Notation. Let $(A,\mathfrak{m})$ be a local ring, let I be an ideal of A, and let M be a finitely generated A-module of dimension d. We consider the associated graded ring

$$G_I(A) := \bigoplus_{n\geq 0} I^n/I^{n+1},$$

and the associated graded module

$$G_I(M) := \bigoplus_{n\geq 0} I^nM/I^{n+1}M.$$

Given $g \in M\setminus\{0\}$, let n be the largest number such that $g \in I^nM$, and define the initial form of g, denoted g^*, by

$$g^* := g \quad \text{modulo} \quad I^{n+1}M \in I^nM/I^{n+1}M \subseteq G_I(M).$$

If $g=0$, we define $g^*=0$. For an A-submodule N of M,

$$G_I(N,M) := \bigoplus_{n\geq 0}((N\cap I^nM)+I^{n+1}M)/I^{n+1}M$$

will denote the $G_I(A)$-submodule of $G_I(M)$ generated by the initial forms of all elements of N.

If the length $\lambda(M/IM)$ is finite, then for sufficiently large values of n, $\lambda(M/I^nM)$ is a polynomial $P_I^M(n)$ in n of degree d, the Hilbert polynomial of (I,M). We write this polynomial in terms of binomial coefficients:

$$P_I^M(n) = e_0(I,M)\binom{n+d-1}{d} - e_1(I,M)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I,M).$$

The coefficients $e_i(I,M)$ are integers and we call them the Hilbert coefficients of (I,M).

1.2. The (S_2) property of Serre. If A is a Noetherian ring, we say that a finitely generated A-module M satisfies Serre's (S_2) property if for every prime ideal $\mathfrak{p}$ of A,

$$\operatorname{depth} M_{\mathfrak{p}} \geq \inf\{2, \dim M_{\mathfrak{p}}\}.$$

We say that the ring A satisfies (S_2) if it satisfies (S_2) as an A-module, i.e., A has no embedded prime ideals and $\operatorname{ht}\mathfrak{p}=1$ for all $\mathfrak{p}\in \operatorname{Ass}(A/xA)$ for any regular element $x\in A$.

We recall the definition of the S_2-ification of a Noetherian domain.

1.3. Definition. Let A be a Noetherian domain. We say that a domain B is an S_2-ification of A if

(1) $A \subseteq B \subseteq Q(A)$ and B is module-finite over A,
(2) B is (S_2) as an A-module, and

(3) for all b in $B\setminus A$, $\operatorname{ht} D(b)\geq 2$, where $D(b)=\{a\in A\mid ab\in A\}$.

1.4. *Remark.* ([17, 2.4]) Set $C:=\{b\in Q(A)\mid \operatorname{ht} D(b)\geq 2\}$. Then A has an S_2-ification if and only if C is a finite extension of A, in which case $\tilde{A}=C$. It is also easy to observe that $\widetilde{A}$ is a finite extension of A inside the quotient field, minimal with the property that it has the (S_2) property as an A-module.

1.5. *Remark.* The S_2-ification does exist for a large class of Noetherian domains. For instance, if A is a universally catenary, analytically unramified domain, then A has an S_2-ification ([12, EGA,5.11.2]). Also, for any local domain $(A,\mathfrak{m})$ that has a canonical module ω, $A\hookrightarrow \operatorname{Hom}_A(\omega,\omega)$ is an S_2-ification of A ([17, 2.7]).

We refer to [12], [1], [2], and [17] for more results about S_2-ification.

1.6. First coefficient ideals. Shah ([19, Theorem 1]) has proved that if I is an ideal primary to the maximal ideal of a formally equidimensional local ring $(A,\mathfrak{m})$, then the set

$$\{J\mid J \text{ ideal of } A, J\supseteq I, e_i(I,A)=e_i(J,A) \text{ for } i=0,1\}$$

has a unique maximal element $I_{\{1\}}$, the *first coefficient ideal* of I. For more about the structure and properties of first coefficient ideals we refer the reader to the original paper of Shah [19] and the series of papers of Heinzer, Lantz, Johnston, and Shah ([13], [14], [15]).

In [7] we have proved the following result:

1.7. Theorem ([7] Theorem 2.5 and Lemma 2.4)**.** *Let $(A,\mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and positive dimension, and let I be an $\mathfrak{m}$-primary ideal of A. Let $\widetilde{S}=\bigoplus_{n\in\mathbb{Z}} I_n t^n$ be the S_2-ification of $S=A[It,t^{-1}]$. Then*

$$I_n\cap A=(I^n)_{\{1\}} \quad \textit{for all } n\geq 1.$$

If A has the (S_2) property, then I_n is an ideal of A, hence $I_n=(I^n)_{\{1\}}$ for all $n\geq 1$.

1.8. Hilbert functions of bigraded modules. We first introduce some known facts about Hilbert functions of bigraded modules. For a detailed description of their properties and complete proofs we refer the reader to [8], [20], and [21] (in these papers the theory is developed for bigraded rings but it can be easily extended to bigraded modules).

Let $R=\bigoplus_{i,j=0}^{\infty} R_{ij}$ be a bigraded ring and let $T=\oplus_{i,j=0}^{\infty} T_{ij}$ be a bigraded R-module. Assume that R_{00} is Artinian and that R is finitely generated as an R_{00}-algebra by elements of R_{01} and R_{10}. The Hilbert function of T is defined to be

$$h_T(i,j)=\lambda_{R_{00}}(T_{i,j}).$$

For i,j sufficiently large, the function $h_T(i,j)$ becomes a polynomial $p_T(i,j)$. If d denotes the dimension of the module T, we can write this polynomial in the form

$$p_T(i,j)=\sum_{\substack{k,l\geq 0\\ k+l\leq d-2}} a_{k,l}(T)\binom{i+k}{k}\binom{j+l}{l},$$

with $a_{k,l}(T)$ integers and $a_{k,d-k-2}(T) \geq 0$.

We also consider the sum transform of h_T with respect to the first variable defined by

$$h_T^{(1,0)}(i,j) = \sum_{u=0}^{i} h_T(u,j),$$

and the sum transform of $h_T^{(1,0)}$ with respect to the second variable,

$$h_T^{(1,1)}(i,j) = \sum_{v=0}^{j} h_T^{(1,0)}(i,v) = \sum_{v=0}^{j}\sum_{u=0}^{i} h(u,v).$$

For i,j sufficiently large, $h^{(1,0)}(i,j)$ and $h^{(1,1)}(i,j)$ become polynomials with rational coefficients of degrees at most $d-1$ and d respectively. As usual, we can write these polynomials in terms of binomial coefficients

$$p_T^{(1,0)}(i,j) = \sum_{\substack{k,l\geq 0\\ k+l\leq d-1}} a_{k,l}^{(1,0)}(T)\binom{i+k}{k}\binom{j+l}{l},$$

with $a_{k,l}^{(1,0)}(T)$ integers and $a_{k,d-k-1}^{(1,0)}(T) \geq 0$, and

$$p_T^{(1,1)}(i,j) = \sum_{\substack{k,l\geq 0\\ k+l\leq d}} a_{k,l}^{(1,1)}(T)\binom{i+k}{k}\binom{j+l}{l},$$

with $a_{k,l}^{(1,1)}(T)$ integers and $a_{k,d-k}^{(1,1)}(T) \geq 0$.

Since

$$h_T(i,j) = h_T^{(1,0)}(i,j) - h_T^{(1,0)}(i-1,j),$$

we get

$$a_{k+1,l}^{(1,0)}(T) = a_{k,l}(T) \quad \text{for } k,l \geq 0, k+l \leq d-2. \tag{1.8.1}$$

Similarly we have

$$h_T^{(1,0)}(i,j) = h_T^{(1,1)}(i,j) - h_T^{(1,1)}(i,j-1),$$

which implies that

$$a_{k,l+1}^{(1,1)}(T) = a_{k,l}^{(1,0)}(T) \quad \text{for } k,l \geq 0, k+l \leq d-1. \tag{1.8.2}$$

2. GENERALIZED HILBERT COEFFICIENTS

In this section we define Hilbert coefficients for an arbitrary ideal I in a local ring $(A,\mathfrak{m})$. The k^{th} generalized Hilbert coefficient $j_k(I)$ is an element of $\mathbb{Z}^{k+1}$ whose first k components are 0 when the ideal I is primary to the maximal ideal $\mathfrak{m}$. We also show that sufficiently general hyperplane sections behave well with respect to the generalized Hilbert coefficients. This is one of the main properties that one would expect from a "good" definition of these coefficients.

Let $(A,\mathfrak{m})$ be a local ring, let I be an ideal of A, and let M be a finitely generated A-module of dimension d. Consider the bigraded ring $R = G_{\mathfrak{m}}(G_I(A))$ and the bigraded R-module $T = G_{\mathfrak{m}}(G_I(M))$, where the graded components are

$$R_{ij} = (\mathfrak{m}^i I^j + I^{j+1})/(\mathfrak{m}^{i+1} I^j + I^{j+1}) \quad \text{and}$$
$$T_{ij} = (\mathfrak{m}^i I^j M + I^{j+1}M)/(\mathfrak{m}^{i+1} I^j M + I^{j+1}M), \text{ respectively.}$$

Observe that $R_{00} = A/\mathfrak{m}$ and $\dim T = \dim M = d$.

As described in 1.8, we define the polynomials $p_R^{(1,0)}(i,j)$, $p_R^{(1,1)}(i,j)$, $p_T^{(1,0)}(i,j)$, and $p_T^{(1,0)}(i,j)$. Note that for $i,j \gg 0$

$$p_R^{(1,0)}(i,j) = \lambda\left(I^j/(\mathfrak{m}^{i+1} I^j + I^{j+1})\right) \quad \text{and}$$
$$p_T^{(1,0)}(i,j) = \lambda\left(I^j M/(\mathfrak{m}^{i+1} I^j M + I^{j+1}M)\right).$$

2.1. Definition. Let $(A,\mathfrak{m})$ be a local ring, let I be an ideal of A, and let M be a finitely generated A-module. Using the notation introduced in 1.8, we define

$$j_k(I,M) := (a_{k,d-k}^{(1,1)}(T), a_{k-1,d-k}^{(1,1)}(T), \dots, a_{0,d-k}^{(1,1)}(T)) \in \mathbb{Z}^{k+1} \quad \text{for } 0 \le k \le d,$$

and call them the *generalized Hilbert coefficients* of (I,M).

Our main concern will be with the first two coefficients

$$j_0(I,M) = a_{0,d}^{(1,1)}(T) \quad \text{and}$$
$$j_1(I,M) = (a_{1,d-1}^{(1,1)}(T), a_{0,d-1}^{(1,1)}(T)).$$

To simplify the notation, we denote $j_1(I,M) = (j_1^1(I,M), j_1^2(I,M))$.

2.2. *Remark.* We also have

$$j_0(I,M) = a_{0,d-1}^{(1,0)}(T),$$
$$j_1(I,M) = (a_{1,d-2}^{(1,0)}(T), a_{0,d-2}^{(1,0)}(T)),$$
$$\dots$$
$$j_{d-1}(I,M) = (a_{d-1,0}^{(1,0)}(T), a_{d-2,0}^{(1,0)}(T), \dots, a_{0,0}^{(1,0)}(T)).$$

This follows from the equalities (1.8.1) and (1.8.2). Note that we need to assume $d = \dim M \ge 2$ in order to refer to $j_1(I,M)$ as $(a_{1,d-2}^{(1,0)}(T), a_{0,d-2}^{(1,0)}(T))$. For technical reasons (see Proposition 2.11), we will prefer this interpretation of the generalized Hilbert coefficients. (We only need $d = \dim M \ge 1$ in order to see $j_1(I,M)$ as $(a_{1,d-1}^{(1,1)}(T), a_{0,d-1}^{(1,1)}(T))$.)

2.3. *Remark.* The coefficients we defined are a generalization of the classical Hilbert coefficients. Indeed, when I is $\mathfrak{m}$-primary,

$$j_k(I,M) = (0,0,\dots,0,(-1)^k e_k(I,M)) \in \mathbb{Z}^{k+1} \quad \text{for } 0 \le k \le d,$$

where the first k components are 0 and $e_k(I,M)$ is the k^{th} Hilbert coefficient of (I,M). To see this, note that if I is $\mathfrak{m}$-primary, there exists t such that $\mathfrak{m}^t \subseteq I$, and then, for i,j large enough,

$$p_T^{(1,1)}(i,j) = \lambda(M/I^{j+1}M).$$

An elementary identification of the coefficients gives the above equalities.

2.4. j-multiplicities. Achilles and Manaresi [3] defined a multiplicity for ideals of maximal analytic spread that generalizes the classical Samuel multiplicity. For a detailed presentation of this multiplicity we refer the reader to [9, Chap. 6].

Let $(A,\mathfrak{m})$ be a local ring, let I be an ideal, and let M be a finitely generated A-module. Then $H^0_{\mathfrak{m}}(G_I(M))$ is a graded $G_I(A)$-submodule of $G_I(M)$ and is annihilated by $\mathfrak{m}^k$ for k large enough, so it may be considered as a module over $\overline{G_I}(A) := G_I(A) \otimes_A A/\mathfrak{m}^k$. Then $e(H^0_{\mathfrak{m}}(G_I(M))) := e(\overline{G_I}(A)^+, H^0_{\mathfrak{m}}(G_I(M)))$ is well defined, where $\overline{G_I}(A)^+$ denotes the ideal of $\overline{G_I}(A)$ of elements of positive degree. Thus we can define

$$j(I,M) := \begin{cases} e(H^0_{\mathfrak{m}}(G_I(M)) & \text{if } \dim M = \dim H^0_{\mathfrak{m}}(G_I(M)) \\ 0 & \text{if } \dim M > \dim H^0_{\mathfrak{m}}(G_I(M)) \end{cases}$$

Note that $j(I,M) \neq 0$ if and only if $\ell_M(I) = \dim M$ [9, 6.1.6(1)], where $\ell_M(I) = \dim G_I(M)/\mathfrak{m}G_I(M)$ (the analytic spread of I in M).

2.5. Generalized Samuel multiplicity. In [4] Achilles and Manaresi defined another generalization of the Samuel multiplicity. Our presentation will be given in the slightly more general context of modules.

Let I be an arbitrary ideal in a local ring $(A,\mathfrak{m})$, and let M be a finitely generated A-module. Using the notation introduced in 1.8, denote

$$c_i(I,M) := a^{(1,1)}_{i,d-i}(T) \quad (0 \leq i \leq d),$$

where $T = G_{\mathfrak{m}}(G_I(M))$. The sequence $(c_i(I,M))_{0\leq i\leq d}$ is called the multiplicity sequence of (I,M). In the case $M = A$ we simply denote $c_i = c_i(I,A)$.

Note that this sequence consists of the leading coefficients of the generalized Hilbert coefficients that we defined in 1.8.

We state the following proposition proved in [4] (we present a version for modules).

2.6. Proposition ([4, Proposition 2.3])**.** *Let $(A,\mathfrak{m})$ be a local ring, let I be a proper ideal of A, and let M be a finitely generated A-module. Set $l = \dim G_I(M)/\mathfrak{m}G_I(M)$ and $q = \dim(M/IM)$. Then*

(i) *$c_k(I,M) = 0$ for $k < d-l$ or $k > q$;*
(ii) *$c_{d-l}(I,M) = \sum_{\beta} e(\mathfrak{m}G_\beta, G_I(M)_\beta)e(G/\beta)$, where β runs through the all highest dimensional associated primes of $G_I(M)/\mathfrak{m}G_I(M)$ such that $\dim(G/\beta) + \dim G_\beta = \dim G$;*
(iii) *$c_q(I,M) = \sum_{\mathfrak{p}} e(IA_{\mathfrak{p}}, M_{\mathfrak{p}})e(A/\mathfrak{p})$, where $\mathfrak{p}$ runs through the all highest dimensional associated primes of M/IM such that $\dim A/\mathfrak{p} + \dim A_{\mathfrak{p}} = \dim A$.*

Achilles and Manaresi [4, Proposition 2.4] also proved that the j-multiplicity $j(I,M)$ is equal to the coefficient $c_0(I,M)$. For more details we refer the reader to the original paper of Achilles and Manaresi [4] (the proofs can be immediately extended to the version for modules we present here).

We will prove that the multiplicity sequence defined above is an invariant of the ideal up to its integral closure. If $J \subseteq I$, we say that J is a reduction of (I,M) if there exists n such that $JI^nM = I^{n+1}M$.

2.7. Proposition. *Let $(A,\mathfrak{m})$ be a local ring, let $J \subseteq I$ be proper ideals of A, and let M be a finitely generated A-module. If J is a reduction of (I,M), then $c_i(J,M) = c_i(I,M)$ for $i = 0,\dots,d$.*

Since the proof requires technical results that will be made clear later, we postpone it until the end of this paper.

Before proceeding further, we need to introduce more notation.

If x is an element of A, denote by x' the initial form of $x^* \in G_I(A)$ in $R = G_\mathfrak{m}(G_I(A))$. Similarly, if J is an ideal in A, let

$$J' = G_\mathfrak{m}(G_I(J,A),G_I(A)) \subseteq R$$

be the ideal generated by all x' when $x \in J$, and if N is an A-submodule of M, we denote

$$N' = G_\mathfrak{m}(G_I(N,A),G_I(M)) \subseteq T = G_\mathfrak{m}(G_I(M)).$$

2.8. Definition ([8]). Let $R = G_\mathfrak{m}(G_I(A))$ and let $(0) = N_1 \cap N_2 \cap \dots \cap N_r \cap N_{r+1} \cap \dots \cap N_t$ be an irredundant primary decomposition of (0) in the R-module $T = G_\mathfrak{m}(G_I(M))$. Denote $P_i = \sqrt{(N_i :_R T)}$, $i = 1,\dots,t$. Assume that

$$I' \subseteq P_{r+1},\dots,P_t \quad \text{and} \tag{2.8.1}$$

$$I' \not\subseteq P_1,\dots,P_r. \tag{2.8.2}$$

We say that $x \in I$ is a superficial element for (I,M) if $x' \notin P_1,\dots,P_r$.

Note that we can always choose $x \in I \setminus \mathfrak{m}I$ superficial element for (I,M).

2.9. *Remark.* Let $x \in I$ be a superficial element for (I,M). By (2.8.1), there exists k such that $(I')^kT \subseteq N_{r+1} \cap \dots \cap N_t$. Then

$$(0' :_T x') = \bigcap_{i=1}^{t} (N_i :_T x') \subseteq N_1 \cap \dots \cap N_r,$$

hence

$$(I')^kT \cap (0' :_T x') \subseteq N_1 \cap N_2 \cap \dots \cap N_r \cap N_{r+1} \cap \dots \cap N_t = (0). \tag{2.9.1}$$

The following lemma, in its version for ideals, is due to Dade [8, 3.1](unpublished thesis). For convenience, we present here a proof.

2.10. Lemma. *Let A be a Noetherian ring, let I be an ideal of A, let M be a finitely generated A-module, and let $L \subseteq K$ be two submodules of M such that the length $\lambda(K/L)$ is finite. Then*

$$\lambda(K/L) = \lambda(G_I(K,M)/G_I(L,M)).$$

Proof. Consider the descending chain of modules

$$\frac{K \cap IM + L}{L} \supseteq \frac{K \cap I^2M + L}{L} \supseteq \dots.$$

The module K/L has finite length, so there exists N such that

$$\frac{K \cap I^nM + L}{L} = \frac{K \cap I^{n+1}M + L}{L} \quad \text{for } n > N$$

which implies that

$$K \cap I^nM + L = K \cap I^{n+1}M + L \quad \text{for } n > N.$$

So, for $n > N$,

$$K \cap I^nM + L \subseteq \bigcap_{k \geq 1} (K \cap I^kM + L) \subseteq \bigcap_{k \geq 1} (I^kM + L) = L,$$

i.e., $K \cap I^nM = L \cap I^nM$.

Finally,

$$\begin{aligned}
\lambda(K/L) &= \lambda\left(\frac{K+IM}{L+IM}\right) + \lambda\left(\frac{K \cap IM}{L \cap IM}\right)\\
&= \lambda\left(\frac{K+IM}{L+IM}\right) + \lambda\left(\frac{K \cap IM + I^2M}{L \cap IM + I^2M}\right) + \lambda\left(\frac{K \cap I^2M}{L \cap I^2M}\right)\\
&\dots\\
&= \lambda\left(\frac{K+IM}{L+IM}\right) + \dots + \lambda\left(\frac{K+I^NM}{L+I^NM}\right)\\
&= \lambda\left(\frac{G_I(K,M)}{G_I(L,M)}\right).
\end{aligned}$$

□

The following proposition shows that sufficiently general hyperplane sections behave well with respect to the generalized Hilbert coefficients.

2.11. Proposition. *Let $(A,\mathfrak{m})$ be a local ring and let M be a finitely generated A-module. Suppose that $x \in I$ is a superficial element for (I,M) and a nonzerodivisor on M with $x' \in R_{01}$. Denote $\overline{T} = G_{\overline{\mathfrak{m}}}(G_{\overline{I}}(\overline{M}))$, where $\overline{A} = A/xA$, $\overline{I} = I \otimes_A \overline{A}$, and $\overline{M} = M \otimes_A \overline{A}$. Then, for i, j large,*

$$h_T^{(1,0)}(i,j) - h_T^{(1,0)}(i,j-1) = h_{\overline{T}}^{(1,0)}(i,j).$$

In particular, $j_0(I,M) = j_0(\overline{I},\overline{M}), j_1(I,M) = j_1(\overline{I},\overline{M}), \dots, j_{d-1}(I,M) = j_{d-1}(\overline{I},\overline{M})$, where d denotes the dimension of the module M.

Proof. The proof relies on Lemma 2.10, a technique also used by Dade in [8].

We have the following exact sequence

$$0 \to K \to \frac{I^jM}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M} \to \frac{I^jM+xM}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M+xM} \to 0,$$

where

$$\begin{aligned} K &= \frac{I^jM\cap(\mathfrak{m}^{i+1}I^jM+I^{j+1}M+xM)}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M} \\ &= \frac{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)+I^jM\cap xM}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M} \\ &\cong \frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM} \quad . \end{aligned}$$

From this exact sequence we get

$$\begin{aligned} h_{\overline{T}}^{(1,0)}(i,j) &= \lambda\Big(\frac{I^jM+xM}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M+xM}\Big) \\ &= \lambda\Big(\frac{I^jM}{\mathfrak{m}^{i+1}I^jM+I^{j+1}M}\Big) - \lambda\Big(\frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM}\Big). \end{aligned}$$

Therefore we need to prove that for $i,j\gg 0$

$$\lambda\Big(\frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM}\Big) = \lambda\Big(\frac{I^{j-1}M}{\mathfrak{m}^{i+1}I^{j-1}M+I^jM}\Big).$$

We have

$$\begin{aligned} \lambda\Big(\frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM}\Big) &= \lambda\Big(\frac{x(I^jM:x)}{x((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)}\Big) \\ &= \lambda\Big(\frac{(I^jM:x)}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x}\Big) \\ &= \lambda\Big(\frac{(I^jM:x)'}{((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'}\Big), \end{aligned}$$

where the last equality follows by a successive application of Lemma 2.10.

By Remark 2.9, there exists c such that $(I')^cT\cap(0':_T x') = (0)$. We claim that for $j>c$

$$(I^jM:x)'\cap(I')^cT = (I^{j-1}M)' \quad \text{and} \tag{2.11.1}$$

$$((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'\cap(I')^cT = (\mathfrak{m}^{i+1}I^{j-1}M+I^jM)'. \tag{2.11.2}$$

We first prove (2.11.1). Let $y \in (I^jM : x)$ such that $0 \neq y' \in (I')^cT$. Since $(I')^cT \cap (0' :_T x') = (0)$, it follows that $y' \notin (0' : x')$, hence $0 \neq (yx)' \in (I^jM)'$. But $(I^jM)'$ is

$$\begin{array}{ccccccccccccc}
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{0,j} & \oplus & T_{0,j+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{1,j} & \oplus & T_{1,j+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & 0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{2,j} & \oplus & T_{2,j+1} & \oplus & \cdots \\
\oplus & & \oplus & & & & \oplus & & \oplus & & \oplus & & \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since $x' \in R_{01}$, we must have $y' \in (I^{j-1}M)'$.

To see (2.11.2), consider $y \in ((\mathfrak{m}^{i+1}I^jM + I^{j+1}M) : x)$ such that $0 \neq y' \in (I')^cT$. By the choice of c, we have $y' \notin (0' : x')$, hence $(yx)' \in (\mathfrak{m}^{i+1}I^jM + I^{j+1}M)'$ and $(yx)' \neq 0$. The homogeneous components of the graded submodule $(\mathfrak{m}^{i+1}I^jM + I^{j+1}M)' \subseteq T$ are represented below:

$$\begin{array}{ccccccccccccc}
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{0,j+1} & \oplus & T_{0,j+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{1,j+1} & \oplus & T_{1,j+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & 0 & \oplus & T_{i,j+1} & \oplus & T_{i,j+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{i+1,j} & \oplus & T_{i+1,j+1} & \oplus & T_{i+1,j+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \oplus & \cdots & \oplus & 0 & \oplus & T_{i+2,j} & \oplus & T_{i+2,j+1} & \oplus & T_{i+2,j+2} & \oplus & \cdots \\
\oplus & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since $x' \in R_{01}$ we get $y' \in (\mathfrak{m}^{i+1}I^{j-1}M + I^jM)'$.

Then we have

$$\begin{aligned}
&\lambda\Big(\frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM}\Big)\\
&=\lambda\Big(\frac{x(I^jM:x)}{x((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)}\Big)\\
&=\lambda\Big(\frac{(I^jM:x)}{((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)}\Big)\\
&=\lambda\Big(\frac{(I^jM:x)'}{((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'}\Big)\\
&=\lambda\Big(\frac{(I^jM:x)'\cap(I')^cT}{((\mathfrak{m}^iI^jM+I^{j+1}M):x)'\cap(I')^cT}\Big)+\lambda\Big(\frac{(I^jM:x)'+(I')^cT}{((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'+(I')^cT}\Big)\\
&=\lambda\Big(\frac{I^{j-1}M}{(\mathfrak{m}^{i+1}I^{j-1}M+I^jM)}\Big)+\lambda\Big(\frac{(I^jM:x)'+(I')^cT}{((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'+(I')^cT}\Big).
\end{aligned}$$

By the Artin-Rees lemma, there exists p such that for $j>p$

$$I^jM\cap xM=I^{j-p}(I^pM\cap xM),$$

i.e.,

$$x(I^jM:_M x)=xI^{j-p}(I^pM:_M x),$$

or

$$(I^jM:_M x)=I^{j-p}(I^pM:_M x).$$

Then, for $j>p+c$, $(I^jM:x)'\subseteq I'^cT$.

On the other hand, we also have

$$\begin{aligned}
((\mathfrak{m}^{i+1}I^jM+I^{j+1}M):x)'&\subseteq(I^jM:x)'\\
&\subseteq(I')^cT\qquad\text{for } j>n+c\quad\text{and all } i.
\end{aligned}$$

We can now conclude that

$$\lambda\Big(\frac{I^jM\cap xM}{(\mathfrak{m}^{i+1}I^jM+I^{j+1}M)\cap xM}\Big)=\lambda\Big(\frac{I^{j-1}M}{\mathfrak{m}^{i+1}I^{j-1}M+I^jM}\Big),$$

which finishes the proof. □

3. First coefficient ideals-the general case

In this section we define the first coefficient ideal $I_{\{1\}}$ of a not necessarily $\mathfrak{m}$-primary ideal I. We then observe that using the new definition of $I_{\{1\}}$, Theorem 1.7 is true in general, without assuming that I is $\mathfrak{m}$-primary.

For reasons that will become obvious later, we need again to introduce the notion in the more general context of modules.

3.1. Definition. Let M be a finitely generated A module and let I be an ideal of A with $\dim M/IM < \dim M$. We define $I^M_{\{1\}}$, the first coefficient ideal of (I,M), to be the ideal of A

$$I^M_{\{1\}} = \bigcup (I^{n+1}M :_A aM),$$

where the union ranges over all $n \geq 1$ and all $a \in I^n \setminus I^{n+1}$ such that a^* is part of a system of parameters of $G_I(M)$. If $M = A$, we simply denote $I^M_{\{1\}} = I_{\{1\}}$.

3.2. *Remark.* Let us observe that our definition coincides with the one given by Shah in the $\mathfrak{m}$-primary case. Indeed, by the structure theorem for the coefficient ideals proved by Shah ([19, Theorem 2]), we have

$$I_{\{1\}} = \bigcup (I^{n+1} :_A a), \tag{3.2.1}$$

where the union ranges over all $n \geq 1$ and all a extendable to some minimal reduction of I^n.

On the other hand, a is extendable to some minimal reduction of I^n if and only if the image of a^* in $G_I(A)/\mathfrak{m}G_I(A)$ is part of a system of parameters. But if the ideal I is $\mathfrak{m}$-primary this is equivalent to the fact that a^* is part of a system of parameters of $G_I(A)$, for the ideal $\mathfrak{m}G_I(A)$ is nilpotent.

Heinzer, Johnston, Lantz, and Shah [14, Theorem 3.17] gave a description of the coefficient ideals involving the blow-up of I. We present here their result for the case of the first coefficient ideals.

The blow-up $\mathcal{B}(I)$ of an ideal I in a local domain A is defined to be the model

$$\mathcal{B}(I) = \{A[I/x]_{\mathfrak{p}} \mid 0 \neq x \in I \text{ and } \mathfrak{p} \in \operatorname{Spec}(A[I/x])\}.$$

$\mathcal{B}(I)$ is the set of all local rings between A and the quotient field $Q(A)$ minimal with respect to domination among those in which the extension of I is a principal ideal. Let $\mathcal{D}_1$ denote the intersection of the local domains on the blow-up $\mathcal{B}(I)$ of dimension at most 1 in which the maximal ideal is minimal over the extension of I (see [13, Definition 3.2]). The main result of [14](Theorem 3.17) says that if A is a formally equidimensional, analytically unramified local domain with infinite residue field and $\dim A > 0$, and I is an $\mathfrak{m}$-primary ideal, then

$$I_{\{1\}} = I\mathcal{D}_1 \cap A. \tag{3.2.2}$$

In a subsequent paper, Heinzer and Lantz [15, 2] prove directly the equivalence of the description of the first coefficient ideals given initially by Shah (see 3.2.1) and the description given by 3.2.2. The argument assumes that the ideal I is $\mathfrak{m}$-primary, but a careful examination of their proof actually shows the following:

3.3. Proposition. *Let $(A,\mathfrak{m})$ be a formally equidimensional local ring of positive dimension, and let I be an arbitrary ideal of A. Then*

$$I\mathcal{D}_1 \cap A = \bigcup (I^{n+1} :_A a),$$

where the union ranges over all $n \geq 1$ and all $a \in I^n \setminus I^{n+1}$ such that a^* is part of a system of parameters of $G_I(A)$.

Note that the right hand side of this equality is exactly the definition of the first coefficient ideals in the general case (see Definition 3.1).

In [7] we have proved Theorem 1.7. The statement of the theorem assumes that I is an $\mathfrak{m}$-primary ideal, but all is used in the proof is that $I_{\{1\}} = I\mathcal{D}_1 \cap A$. Therefore, by the above discussion, we have the following theorem.

3.4. Theorem. *Let $(A,\mathfrak{m})$ be a formally equidimensional, analytically unramified local domain with infinite residue field and positive dimension, and let I be an arbitrary ideal of A. If $\widetilde{S} = \bigoplus_{n\in\mathbb{Z}} I_n t^n$ is the S_2-ification of $S = A[It, t^{-1}]$, then*

$$I_n \cap A = (I^n)_{\{1\}} \quad \text{for all } n \geq 1,$$

where for an ideal J, $J_{\{1\}}$ denotes the first coefficient ideal of J as defined in 3.1.

In particular, if A has the (S_2) property, then $I_n = (I^n)_{\{1\}}$ for all $n \geq 1$.

In this way, the problem of giving a numerical characterization of the S_2-ification of the extended Rees algebra $S = A[It, t^{-1}]$ reduces to the problem of finding a numerical characterization of the generalized first coefficient ideals (Definition 3.1).

The following proposition shows that the union involved in Definition 3.1 can be replaced by a single colon ideal. It is the analogue of Theorem 3 of [19].

Recall that a finitely generated module M over a local ring A is called equidimensional if for every minimal prime ideal $\mathfrak{p}$ of M the module $M/\mathfrak{p}M$ has dimension $\dim M$. We also say that M is formally equidimensional if $\widehat{M}$ (the completion of M in the $\mathfrak{m}$-adic topology) is equidimensional as an $\widehat{A}$-module. If the ring A is complete and M is equidimensional, then $G_I(M)$ is also equidimensional (see [16, 18.24] and [6, 4.5.6]).

3.5. Proposition. *Let M be a finitely generated formally equidimensional A-module and let I be an ideal of A such that $\dim M/IM < \dim M$. Then there exist a fixed integer m and a fixed element x of $I^m \setminus I^{m+1}$ with x^* part of system of parameters of $G_I(M)$ such that*

$$I^M_{\{1\}} = (I^{m+1}M :_A xM).$$

Proof. We can assume that A is complete and that M is equidimensional. Let N be the $G_I(A)$-submodule of $G_I(M)$ generated by $I^M_{\{1\}}M/IM$. By definition, each generator of N is annihilated by a homogeneous element of $G_I(A)$ which is part of a system of parameters of $G_I(M)$. By prime avoidance, we can find a homogeneous element $x^* \in I^m/I^{m+1}$ ($x \in I^m$) that annihilates the entire submodule N and which avoids all the minimal primes in the support of $G_I(M)$. The observation that $G_I(M)$ is equidimensional (implied by the hypothesis) concludes the proof. $\square$

3.6. Proposition. *Let M be a formally equidimensional A-module, and let $I \subseteq J$ be ideals of A such that $\dim M/IM < \dim M$. Then $J \subseteq I^M_{\{1\}}$ if and only if*

$$\dim \bigoplus_{n\geq 0} JI^nM/I^{n+1}M < \dim G_I(M) = \dim M.$$

Proof. Indeed, if we denote $L = \bigoplus_{n\geq 0} JI^nM/I^{n+1}M$, then, by Proposition 3.5, it follows that L is annihilated by an element which is part of a system of parameters of $G_I(M)$. □

3.7. *Remark.* If M is faithful (i.e. $\operatorname{Ann} M = 0$) and J is a (minimal) reduction of (I,M), then J is a (minimal) reduction of I. Indeed, if $I^{n+1}M = JI^nM$ for some n, then, by the determinant trick, J and I have the same integral closure, i.e., J is a reduction of I.

In the $\mathfrak{m}$-primary case it is obvious that the ideal I is a reduction of its first coefficient ideal (by definition). This is still true in the general case, as the following proposition shows.

3.8. Proposition. *Let $(A,\mathfrak{m})$ be a local ring, let M be a finitely generated formally equidimensional A-module, and let I be an ideal of A such that* $\dim M/IM < \dim M$. *If $I \subseteq J \subseteq I^M_{\{1\}}$, then I is a reduction of (J,M).*

Proof. As usual, we may assume that A is a complete local ring. First we prove the proposition in the case when M is faithful. Note that in this case both A and M will be equidimensional, therefore both $G_I(A)$ and $G_I(M)$ are equidimensional of dimension equal to $\dim A = \dim M$ (this is implicitly proved in Theorem 4.5.6 of [6]).

Let us observe that for a faithful A-module M, $\operatorname{Ann} G_I(M)$ is a nilpotent ideal of $G_I(A)$. Indeed, if $\bar{x} \in I^n/I^{n+1}$ is an element of $G_I(A)$ that annihilates $G_I(M)$, then $xM \subseteq I^{n+1}M$, which by the determinant trick implies that $x \in \overline{I^{n+1}}$ (here $\overline{J}$ denotes the integral closure of the ideal J) . If we write the equation of integral dependence we get

$$x^k + a_1x^{k-1} + ... + a_k = 0,$$

with $a_i \in I^{(n+1)i}$. Thus $x^k = -(a_1x^{k-1} + ... + a_k) \in I^{kn+1}$, which implies that $\bar{x} \in G_I(A)$ is nilpotent.

By Proposition 3.5, there exist a fixed integer m and a fixed element $a \in I^m \setminus I^{m+1}$ with $a^* \in G_I(A)$ part of a system of parameters of $G_I(M)$ such that $I^M_{\{1\}} = (I^{m+1}M :_A aM)$. Let $y \in (I^{m+1}M :_A aM)$. Then $yaM \subseteq I^{m+1}M$, and using the determinant trick we get

$$ya \in \overline{I^{m+1}}. \tag{3.8.1}$$

Since $\operatorname{Ann}(G_I(M))$ is nilpotent and $G_I(A)$ is equidimensional, a^* is part of a system of parameters of $G_I(A)$, i.e. $at^m \in S = A[It,t^{-1}]$ is not contained in any minimal prime divisor of $t^{-1}S$.

We claim that from the above assertion and (3.8.1) it follows that $y \in \bar{I}$. To prove this, note that we may also assume that A is a reduced ring. Let $\overline{T} = \bigoplus_{n\geq 0} \overline{I^n\overline{A}}t^n$ be the integral closure of T in its total quotient ring. Since the ring $\overline{A}$ is equidimensional (it is a local catenary ring satisfying the (S_2) property; see [12, 5.10.9]), the ring $\overline{T}/t^{-1}\overline{T}$ is also equidimensional (implicitly proved in Theorem 4.5.6 of [6]; note that $(\overline{I^n})_{n\geq 0}$ is a Noetherian filtration) and is a finite extension of $T/t^{-1}T$. In particular, any minimal prime of $t^{-1}\overline{T}$ contracts back to a minimal prime of $t^{-1}T$. Thus the image of at^m does not belong to any associate prime of $t^{-1}\overline{T}$, hence a^* is a nonzerodivisor on $\overline{T}/t^{-1}\overline{T}$. By (3.8.1) we get $y \in \overline{I\overline{A}} \cap A = \overline{I}$. □

4. THE MAIN RESULT

We now prove two propositions that will be the main tools for proving Theorem 4.5 in dimension 2.

4.1. Proposition. *Let M be a finitely generated formally equidimensional A-module of dimension 2, and let $I \subseteq J$ be two ideals of A such that $\dim M/IM < \dim M$. If $J \subseteq I^M_{\{1\}}$, then there exist positive integers k and l such that*

$$\mathfrak{m}^k I^j M \subseteq J^j M \quad \text{for } j \geq l.$$

In particular,

$$\lambda(J^j M/I^j M) < \infty \quad \text{for } j \gg 0.$$

Proof. Denote by N the $G = G_I(A)$-submodule of $G_I(M)$ generated in degree 0 by JM/IM, i.e.

$$N = \bigoplus_{n \geq 0} JI^n M/I^{n+1}M.$$

By Proposition 3.6, we have $\dim_G(N) \leq \dim G(M) - 1 = 1$, which implies that $\dim_{G_\mathfrak{m}(G)} G_\mathfrak{m}(N) \leq 1$. Since

$$G_\mathfrak{m}(N) = \bigoplus_{i,j \geq 0} \frac{\mathfrak{m}^i JI^j M + I^{j+1}M}{\mathfrak{m}^{i+1} JI^j M + I^{j+1}M},$$

It follows that for $i, j \gg 0$

$$\lambda\Big(\frac{\mathfrak{m}^i JI^j M + I^{j+1}M}{\mathfrak{m}^{i+1} JI^j M + I^{j+1}M}\Big)$$

is a polynomial of degree $\leq \dim G_\mathfrak{m}(N) - 2 \leq -1$, so there exist i_0, j_0 such that

$$\lambda\Big(\frac{\mathfrak{m}^i JI^j M + I^{j+1}M}{\mathfrak{m}^{i+1} JI^j M + I^{j+1}M}\Big) = 0 \quad \text{for } i \geq i_0, j \geq j_0.$$

By Nakayama's lemma we then obtain

$$\mathfrak{m}^i JI^j M \subseteq I^{j+1}M \quad \text{for } i \geq i_0, j \geq j_0. \tag{4.1.1}$$

Since I is a reduction of (J, M) (3.8) there exists n such that $I^j J^n M = J^{n+j} M$ for $j \geq 1$. By (4.1.1) it follows that

$$\mathfrak{m}^{ni} J^n I^j M \subseteq I^{n+j} M \quad \text{for } i \geq i_0, j \geq j_0,$$

which in conjunction with the previous equality implies that

$$\mathfrak{m}^{ni} J^{n+j} M \subseteq I^{n+j} M \quad \text{for } i \geq i_0, j \geq j_0.$$

Take $k = i_0$ and $l = n + j_0$. □

4.2. Proposition. *Let $(A,\mathfrak{m})$ be a local ring and let M be a finitely generated formally equidimensional A-module of dimension ≤ 2. Consider $I \subseteq J$ two ideals in A with* $\dim M/IM < \dim M$ *such that $I \subseteq J \subseteq I^M_{\{1\}}$. Then, for i,j large enough,*

1) $\lambda(J^jM/I^jM)$ *is a constant;*
2) $\lambda(\mathfrak{m}^{i+1}J^jM + J^{j+1}M/\mathfrak{m}^{i+1}I^jM + I^{j+1}M)$ *is a constant;*
3) $\lambda(I^jM/\mathfrak{m}^{i+1}I^jM + I^{j+1}M) = \lambda(J^jM/\mathfrak{m}^{i+1}J^jM + J^{j+1}M)$.

Proof. By Proposition 4.1, $\lambda(J^jM/I^jM)$ is finite for j large enough, so for the first part of the proposition we can use an argument similar (but in module version) to the one used by Shah in the proof of Theorem 2 of [19].

Since $I \subseteq J \subseteq I^M_{\{1\}}$, I is a reduction of (J,M) (see Proposition 3.8), hence there exists an integer s such that $I^nJ^sM = J^{n+s}M$ for all n. Then we have

$$\begin{aligned}
\lambda\left(J^{s+n}M/I^{s+n}M\right) &= \lambda\left(J^sI^nM/I^{s+n}M\right)\\
&= \sum_{i=1}^{s} \lambda\left(J^iI^{n+s-i}M/J^{i-1}I^{n+s-i+1}M\right)\\
&= \sum_{i=1}^{s} \lambda\left(J^{i-1}I^{s-i}JI^nM/J^{i-1}I^{s-i}I^{n+1}M\right)\\
&\leq \sum_{i=1}^{s} c_i\lambda\left(JI^nM/I^{n+1}M\right)
\end{aligned}$$

where c_i is the number of generators of $J^{i-1}I^{s-i}M$. Set $c = \sum c_i$. Then

$$\lambda\left(J^{s+n}M/I^{s+n}M\right) \leq c\lambda\left(JI^nM/I^{n+1}M\right).$$

On the other hand, by Proposition 3.6, for n large enough, $\lambda(JI^nM/I^{n+1}M)$ is a polynomial of degree $\leq \dim M - 2$, so it must be a constant ($\dim M \leq 2$). Thus $\lambda(J^jM/I^jM)$ is a constant for $j \gg 0$.

For the second part, let us observe that

$$\begin{aligned}
&\lambda\left(\frac{J^jM}{I^jM}\right) - \lambda\left(\frac{\mathfrak{m}^{i+1}J^jM + J^{j+1}M}{\mathfrak{m}^{i+1}I^jM + I^{j+1}M}\right)\\
&= \lambda\left(\frac{J^jM}{\mathfrak{m}^{i+1}J^jM + J^{j+1}M}\right) - \lambda\left(\frac{I^jM}{\mathfrak{m}^{i+1}I^jM + I^{j+1}M}\right)\\
&= [j_1^1(J,M) - j_1^1(I,M)]i + [j_0(J,M) - j_0(I,M)]j + j_1^2(J,M) - j_1^2(I,M).
\end{aligned}$$

By 2.6, it follows that

$$j_0(J,M) = j_0(I,M) \text{ and } j_1^1(J,M) = j_1^1(I,M),$$

and therefore the last expression is a constant. Using the first part we can now conclude the second part.

By Lemma 4.1 and Lemma 2.10, we have

$$\begin{aligned}\lambda(J^{j+1}M/I^{j+1}M) &= \lambda\Big(G_{\mathfrak m}(J^{j+1}M)/G_{\mathfrak m}(I^{j+1}M)\Big)\\ &= \lambda\Big(\bigoplus_{k\geq 0}\frac{(J^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}{(I^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}\Big)\\ &= \lambda\Big(\bigoplus_{k=0}^{t}\frac{(J^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}{(I^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}\Big)\end{aligned}$$

for some fixed integer t independent of j (by part (1) we can do this). Similarly,

$$\begin{aligned}\lambda\Big(\frac{\mathfrak m^{i+1}J^jM+J^{j+1}M}{\mathfrak m^{i+1}I^jM+I^{j+1}M}\Big) &= \lambda\Big(\frac{G_{\mathfrak m}(\mathfrak m^{i+1}J^jM+J^{j+1}M)}{G_{\mathfrak m}(\mathfrak m^{i+1}I^jM+I^{j+1}M)}\Big)\\ &= \lambda\Big(\bigoplus_{k=0}^{s}\frac{\big((\mathfrak m^{i+1}J^jM+J^{j+1}M)\cap \mathfrak m^kM\big)+\mathfrak m^{k+1}M}{\big((\mathfrak m^{i+1}I^jM+I^{j+1}M)\cap \mathfrak m^kM\big)+\mathfrak m^{k+1}M}\Big),\end{aligned}$$

for some fixed integer s independent of i and j (we use here the second part of the statement). We may assume $s=t$. On the other hand, for $i\geq t$,

$$(\mathfrak m^{i+1}J^jM+J^{j+1}M)\cap \mathfrak m^kM = \mathfrak m^{i+1}J^jM+(\mathfrak m^kM\cap J^{j+1}M)$$

and

$$(\mathfrak m^{i+1}I^jM+I^{j+1}M)\cap \mathfrak m^kM = \mathfrak m^{i+1}I^jM+(\mathfrak m^kM\cap I^{j+1}M).$$

This implies that

$$\big((\mathfrak m^{i+1}J^jM+J^{j+1}M)\cap \mathfrak m^kM\big)+\mathfrak m^{k+1}M = (\mathfrak m^kM\cap J^{j+1}M)+\mathfrak m^{k+1}M$$

and

$$\big((\mathfrak m^{i+1}I^jM+I^{j+1}M)\cap \mathfrak m^kM\big)+\mathfrak m^{k+1}M = (\mathfrak m^kM\cap I^{j+1}M)+\mathfrak m^{k+1}M.$$

We then get

$$\begin{aligned}\lambda\Big(\frac{\mathfrak m^{i+1}J^jM+J^{j+1}M}{\mathfrak m^{i+1}I^jM+I^{j+1}M}\Big) &= \lambda\Big(\bigoplus_{k=0}^{t}\frac{(J^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}{(I^{j+1}M\cap \mathfrak m^kM)+\mathfrak m^{k+1}M}\Big)\\ &= \lambda(J^{j+1}M/I^{j+1}M)\\ &= \lambda(J^jM/I^jM),\end{aligned}$$

where the last equality follows from part (1). □

4.3. Lemma. *Let $(A,\mathfrak m)$ be a local ring, let $I\subseteq J$ be two ideals in A, and let M be a finitely generated A-module. Let k be a positive integer.*

1) *If I is a reduction of (J,M), then $j_0(J,M)=j_0(J,I^kM)$.*
2) *If I is a reduction of (J,M), then I is a reduction of (J,I^kM).*
3) *Assume that $\dim M/IM<\dim M$ and that M is equidimensional. If I is a reduction of (J,I^kM), then I is a reduction of (J,M) .*
4) *If I is a reduction of (J,M), then $j_1(I,I^kM)=j_1(J,I^kM)$ implies that $j_1(I,M)=j_1(J,M)$.*

5) *$J \subseteq I^{M}_{\{1\}}$ if and only if $J \subseteq I^{I^kM}_{\{1\}}$.*

Proof. (1) I is a reduction of (J,M), so there exists a positive integer n such that $IJ^nM = J^{n+1}M$. So for $j \gg 0$,

$$\lambda\Big(\frac{J^jI^kM}{\mathfrak{m}^{i+1}J^jI^kM + J^{j+1}I^kM}\Big) = \lambda\Big(\frac{J^{j+k}M}{\mathfrak{m}^{i+1}J^{j+k}M + J^{j+1+k}M}\Big), \tag{4.3.1}$$

which implies that $j_0(J,M) = j_0(J,I^kM)$.

(4) also follows from (4.3.1).

(2) is obvious.

(3) Let $\overline{A} = A/\operatorname{Ann} M$, $\overline{I} = I\overline{A}$, and $\overline{J} = J\overline{A}$. Then $\overline{A}$ is an equidimensional ring and $\overline{I}$ is not contained in any minimal prime ideal of $\overline{A}$. Since I is a reduction of (J, I^kM), there exists a positive integer n such that $IJ^nI^kM = J^{n+1}I^kM$. By the determinant trick, it follows that $\overline{I}\overline{J}^n\overline{I}^k = \overline{I}^{k+1}\overline{J}^n$ is a reduction of $\overline{J}^{n+1}\overline{I}^k$, so there exists l such that

$$\overline{I}^{k+1}\overline{J}^n(\overline{J}^{n+1}\overline{I}^k)^l = (\overline{J}^{n+1}\overline{I}^k)^{l+1}.$$

Set $s = kl + k$, $t = nl + n + l$ so that the above equality can be written

$$\overline{I}(\overline{I}^s\overline{J}^t) = \overline{J}(\overline{I}^s\overline{J}^t).$$

We claim that this implies that $\overline{I}$ is a reduction of $\overline{J}$. It is enough to show this after we mod out an arbitrary minimal prime ideal of $\overline{A}$, and since $\overline{I}$ is not contained in any minimal prime ideal of $\overline{A}$, we may therefore assume that $\overline{A}$ is a domain and $\overline{I}$, $\overline{J}$ are nonzero ideals. Using again the determinant trick, we get $\overline{I}$ is a reduction of $\overline{J}$ ($\overline{I}^s\overline{J}^t \neq 0$), which implies that I is a reduction of (J,M).

(5) Denote $K = \bigoplus_{n\geq 0} JI^nM/I^{n+1}M$ and $L = \bigoplus_{n\geq 0} JI^{n+k}M/I^{n+1+k}M$. It is clear that $\dim K = \dim L$. On the other hand, $J \subseteq I^M_{\{1\}}$ if and only if $\dim K < \dim G_I(M)$, and $J \subseteq I^{I^kM}_{\{1\}}$ if and only if $\dim L < \dim G_I(M)$. □

The following proposition shows that the first two generalized Hilbert coefficients are the same up to the first coefficient ideal.

4.4. Proposition. *Let $(A,\mathfrak{m})$ be a local ring, let M be a formally equidimensional A-module, and let I be an ideal of A with $\dim M/IM < \dim M$. If $I \subseteq J \subseteq I^M_{\{1\}}$, then*

$$j_i(I,M) = j_i(J,M) \quad \textit{for } i = 0,1.$$

Proof. We may assume that A is complete and that M is equidimensional.

If $\dim M = 1$, then the conclusion follows from Shah's result (in its version for modules). Indeed, we can replace A by $A/\operatorname{Ann} M$, and then the ideals I and J are primary to the maximal ideal of $A/\operatorname{Ann} M$.

If $\dim M = 2$, from Proposition 4.2 part (3) it follows that for $i, j \gg 0$ we have following equality of polynomial functions of degree one:

$$\lambda(I^jM/\mathfrak{m}^{i+1}I^jM + I^{j+1}M) = \lambda(J^jM/\mathfrak{m}^{i+1}J^jM + J^{j+1}M).$$

By Remark 2.2, it follows that $j_i(I,M) = j_i(J,M)$ for $i = 0,1$.

Assume $\dim M \geq 3$. If $\operatorname{depth}_I M = 0$, replacing M by $I^k M$ for k big enough, we may assume $\operatorname{depth}_I M > 0$ (the previous proposition shows that the hypotheses are preserved).

By Proposition 3.5, there exists an integer $n \geq 1$ and an element $a \in I^n \setminus I^{n+1}$, with a^* part of a system of parameters of $G_I(M)$, such that $I^M_{\{1\}} = (I^{n+1} : aM)$. Since I is a reduction of J (see Proposition 3.8), we can choose $x \in I \setminus \mathfrak{m}J$ superficial element for (J, M) (I' and J' have the same radical in $G_\mathfrak{m}(G_J(A))$). By taking a sufficiently general element, we may also assume that x is a superficial element for (I, M), a^*, x^* are part of a system of parameters of $G_\mathfrak{m}(G_I(A))$, and x is a nonzerodivisor on M ($\operatorname{depth}_I M > 0$).

Denote $\overline{M} = M/xM$. By the choice of x it follows that $I \subseteq J \subseteq I^{\overline{M}}_{\{1\}}$. Indeed, if $y \in J$, then $ya\overline{M} \subseteq I^{n+1}\overline{M}$. But x^* and a^* are part of a system of parameters of $G_I(M)$, so a^* is part of a system of parameters of $G_I(\overline{M}) \cong G_I(M)/x^* G_I(M)$. Then $\bar{y} \in I^{\overline{M}}_{\{1\}}$ and the induction hypothesis gives $j_i(I, \overline{M}) = j_i(J, \overline{M})$ for $i = 0, 1$. Using Proposition 2.11 we now obtain $j_i(I, M) = j_i(J, M)$ for $i = 0, 1$.

Note that we cannot prove the 2-dimensional case by reducing the problem to the 1-dimensional case. The polynomial that gives $\lambda(I, M)(I^j M/\mathfrak{m}^i I^j M + I^{j+1} M)$ for $j \gg 0$ has the form $j_1^1(I, M)i + j_0(I, M)j + j_1^2(I, M)$. By reducing the dimension one more time we would loose the coefficients $j_1^1(I, M)$ and $j_1^2(I, M)$. □

We can now prove the theorem stated in the introduction.

4.5. Theorem. *Let $(A, \mathfrak{m})$ be a local ring, let M be a formally equidimensional A-module, and let $I \subseteq J$ be two ideals of A with $\dim M/IM < \dim M$. The following are equivalent:*

1) $J \subseteq I^M_{\{1\}}$.
2) $j_i(I_\mathfrak{p}, M_\mathfrak{p}) = j_i(J_\mathfrak{p}, M_\mathfrak{p})$ *for $i = 0, 1$ and every* $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proof. The proof of the case $\dim M = 2$ is the crucial part of the argument. Then we can use an induction argument similar to the one used by Flenner and Manaresi in the proof of their theorem (see the introduction).

If $\dim M = 1$, using the same argument used in the proof of the previous theorem, we can reduce the problem to the $\mathfrak{m}$-primary case and Shah's result proves both implications.

As usual, we may assume that $(A, \mathfrak{m})$ is a complete local ring and M is equidimensional. We will prove that for every prime ideal $\mathfrak{p}$, $J_\mathfrak{p} \subseteq (I_\mathfrak{p})^{M_\mathfrak{p}}_{\{1\}}$ and the implication $(1) \Longrightarrow (2)$ will follow from Proposition 4.4.

Let $N = \bigoplus_{n \geq 1} JI^n/I^{n+1}$. Since $J \subseteq I^M_{\{1\}}$, by Remark 3.6, we have $\dim N < \dim G_I(M) = \dim M$. Let $N' = \bigoplus_{n \geq 1} J_\mathfrak{p}(I_\mathfrak{p})^n/(I_\mathfrak{p})^{n+1} = U^{-1}N$, where $U = G^0_I(A) \setminus (\mathfrak{p}/I)$ is a multiplicatively closed subset of $G_I(A)$. Since $G_I(M)$ is equidimensional, we get $\dim N' < \dim U^{-1} G_I(M) = \dim G_{I_\mathfrak{p}}(M_\mathfrak{p})$, i.e. $J_\mathfrak{p} \subseteq (I_\mathfrak{p})^{M_\mathfrak{p}}_{\{1\}}$.

We prove the converse by induction on $d = \dim M$. First assume $\dim M = 2$. We can also assume that M is faithful, so $\dim A = 2$. Since $j_i(I, M) = j_i(J, M)$ for

$i = 0, 1$ there exist i_0, j_0 such that for $i \geq i_0$ and $j \geq j_0$

$$\text{(4.5.1)} \qquad \lambda(I^jM/\mathfrak{m}^{i+1}I^jM + I^{j+1}M) = \lambda(J^jM/\mathfrak{m}^{i+1}J^jM + J^{j+1}M).$$

Let $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$, so by hypothesis $j_i(I_\mathfrak{p}, M_\mathfrak{p}) = j_i(J_\mathfrak{p}, M_\mathfrak{p})$ for $i = 0, 1$. But $\dim A_\mathfrak{p} = 1$, so $I_\mathfrak{p}$ and $J_\mathfrak{p}$ are primary to the maximal ideal. Applying the theory of first coefficient ideals for $\mathfrak{m}$-primary ideals (in a version for modules) we get $\lambda(J_\mathfrak{p}^jM_\mathfrak{p}/I_\mathfrak{p}^jM_\mathfrak{p}) = 0$ for $j \gg 0$ (it is bounded above by a polynomial of degree $\dim A_\mathfrak{p} - 2 = -1$). There are only finitely many elements in $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ that contain I, so there exists $r \geq j_0$ such that for all $\mathfrak{p} \in \operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ and $j \geq r$ we have $\lambda(J_\mathfrak{p}^jM_\mathfrak{p}/I_\mathfrak{p}^jM_\mathfrak{p}) = 0$, and this implies that $\lambda(J^jM/I^jM) < \infty$ for $j \geq r$. Choose $c \geq i_0$ such that $\mathfrak{m}^iJ^rM \subseteq I^rM$ for $i \geq c$.

We are now using an argument similar to the one given in the proof of Proposition 4.2.

For $i \geq c$, we have

$$\begin{aligned}
(*)\,\lambda\Big(\frac{J^rM}{I^rM}\Big) &- \lambda\Big(\frac{\mathfrak{m}^{i+1}J^rM + J^{r+1}M}{\mathfrak{m}^{i+1}I^rM + I^{r+1}M}\Big)\\
&= \lambda\Big(\frac{J^rM}{\mathfrak{m}^{i+1}J^rM + J^{r+1}M}\Big) - \lambda\Big(\frac{I^rM}{\mathfrak{m}^{i+1}I^rM + I^{r+1}M}\Big)\\
&= [j_1^1(J,M) - j_1^1(I,M)]i + [j_0(J,M) - j_0(I,M)]j + j_1^2(J,M) - j_1^2(I,M)\\
&= 0.
\end{aligned}$$

where the last equality follows from hypothesis.

Then, by Lemma 4.1 and Lemma 2.10, we have

$$\begin{aligned}
\lambda(J^{r+1}M/I^{r+1}M) &= \lambda\Big(G_\mathfrak{m}(J^{r+1}M)/G_\mathfrak{m}(I^{r+1}M)\Big)\\
&= \lambda\Big(\bigoplus_{k\geq 0}\frac{(J^{r+1}M \cap \mathfrak{m}^kM) + \mathfrak{m}^{k+1}M}{(I^{r+1}M \cap \mathfrak{m}^kM) + \mathfrak{m}^{k+1}M}\Big)\\
&= \lambda\Big(\bigoplus_{k=0}^{t}\frac{(J^{r+1}M \cap \mathfrak{m}^kM) + \mathfrak{m}^{k+1}M}{(I^{r+1}M \cap \mathfrak{m}^kM) + \mathfrak{m}^{k+1}M}\Big)
\end{aligned}$$

for some fixed integer t (r is fixed).

Using $(*)$ we obtain that for $i \geq c$

$$\begin{aligned}
(**)\,\lambda(J^rM/I^rM) &= \lambda\Big(\frac{\mathfrak{m}^{i+1}J^rM + J^{r+1}M}{\mathfrak{m}^{i+1}I^jM + I^{r+1}M}\Big)\\
&= \lambda\Big(\frac{G_\mathfrak{m}(\mathfrak{m}^{i+1}J^rM + J^{r+1}M)}{G_\mathfrak{m}(\mathfrak{m}^{i+1}I^rM + I^{r+1}M)}\Big)\\
&= \lambda\Big(\bigoplus_{k=0}^{s}\frac{\big((\mathfrak{m}^{i+1}J^rM + J^{j+1}M) \cap \mathfrak{m}^kM\big) + \mathfrak{m}^{k+1}M}{\big((\mathfrak{m}^{i+1}I^rM + I^{r+1}M) \cap \mathfrak{m}^kM\big) + \mathfrak{m}^{k+1}M}\Big)
\end{aligned}$$

for some fixed integer s independent of i (r is fixed). We may assume $s=t$. But for $i \geq c+t$ and $0 \leq k \leq t$ we have

$$(\mathfrak{m}^{i+1}J^jM+J^{r+1}M)\cap\mathfrak{m}^kM=\mathfrak{m}^{i+1}J^rM+(\mathfrak{m}^kM\cap J^{r+1}M)$$

and the similar equality with I instead of J. This implies that for $i \geq c+t$

$$\big((\mathfrak{m}^{i+1}J^jM+J^{r+1}M)\cap\mathfrak{m}^kM\big)+\mathfrak{m}^{k+1}M=(\mathfrak{m}^kM\cap J^{r+1}M)+\mathfrak{m}^{k+1}M$$

and the similar equality for I.

Using the above observations and $(**)$ we have that for $i \geq c+t$

$$\begin{aligned}\lambda(J^rM/I^rM) &= \lambda\Big(\frac{\mathfrak{m}^{i+1}J^rM+J^{r+1}M}{\mathfrak{m}^{i+1}I^rM+I^{r+1}M}\Big)\\ &= \lambda\Big(\bigoplus_{k=0}^{t}\frac{(J^{r+1}M\cap\mathfrak{m}^kM)+\mathfrak{m}^{k+1}M}{(I^{r+1}M\cap\mathfrak{m}^kM)+\mathfrak{m}^{k+1}M}\Big)\\ &= \lambda(J^{r+1}M/I^{r+1}M).\end{aligned}$$

Repeating the argument we conclude that $\lambda(J^jM/I^jM)$ is constant for $j \geq r$.

But this implies that there exists l such that $\mathfrak{m}^lJ^jM \subseteq I^jM$ for all $j \geq r$. Indeed, if L is module of finite length, say l, over a local ring, then $\mathfrak{m}^lL=0$.

So $\mathfrak{m}^iJ^jM \subseteq I^jM$ for all $i \geq l$ and $j \geq r$, hence $\mathfrak{m}^iJI^jM \subseteq \mathfrak{m}^iJ^{j+1}M \subseteq I^{j+1}M$ for all $i \geq l$ and $j \geq r$. Then

$$\lambda\Big(\frac{\mathfrak{m}^iJI^jM+I^{j+1}M}{\mathfrak{m}^{i+1}JI^jM+I^{j+1}M}\Big)=0 \quad \text{for } i \geq l, j \geq r,$$

which implies that $\dim_{G_\mathfrak{m}(G)} G_\mathfrak{m}(N) \leq 1$, where $N=\bigoplus_{n\geq 0} JI^nM/I^{n+1}$. This means that $\dim N \leq 1 = \dim G_I(M)-1$, and by Remark 3.6 we get $J \subseteq I^M_{\{1\}}$.

We now assume that $\dim M \geq 3$. Replacing M by I^kM for a suitable k, we may assume that $\operatorname{depth}_I(M)>0$.

By the theorem of Flenner and Manaresi ([10, Theorem 3.3]; see also the introduction), I is a reduction of (J,M). Then, for a sufficiently general element x in I, we have $j_i(I_\mathfrak{p},M_\mathfrak{p})=j_i(I_\mathfrak{p},\overline{M}_\mathfrak{p})$ and $j_i(J_\mathfrak{p},M_\mathfrak{p})=j_i(J_\mathfrak{p},\overline{M}_\mathfrak{p})$ for $i=0,1$, where $\overline{M}=M/xM$. By the induction hypothesis, we get $J \subseteq I^{\overline{M}}_{\{1\}}$. We still have to prove that $J \subseteq I^M_{\{1\}}$.

Let $K=\bigoplus_{n\geq 0} JI^nM/I^{n+1}M$ and $L=\bigoplus_{n\geq 0}(JI^nM+xM)/(I^{n+1}M+xM)$. Since $J \subseteq I^{\overline{M}}_{\{1\}}$, we have $\dim L < \dim G_I(M/xM)=\dim M/xM=\dim M-1$ (we can choose x to be a nonzerodivisor on M).

For technical reasons (see Proposition 2.11), we will prefer this interpretation of the Generalized Hilbert coefficients.

Consider the exact sequence

$$0 \to U \to K \xrightarrow{\pi} L \to 0 \qquad (4.5.2)$$

where U is the kernel of the canonical epimorphism $K \xrightarrow{\pi} L$.

We have

$$U_n = \frac{JI^nM \cap (I^{n+1}M + xM)}{I^{n+1}M} = \frac{I^{n+1} + (JI^nM \cap xM)}{I^{n+1}M}$$
$$\cong \frac{JI^nM \cap xM}{I^{n+1}M \cap xM} = \frac{x(JI^nM : x)}{x(I^{n+1}M : x)} \cong \frac{JI^nM : x}{I^{n+1}M : x}.$$

On the other hand, for $x \in I$ sufficiently general and $n \gg 0$, $(JI^{n+1}M : x) = JI^nM$ and $(I^{n+1}M : x) = I^nM$.

So, for n large enough, $K_{n-1} \cong U_n$ (isomorphism induced by the multiplication by x), and then the exact sequence 4.5.2 implies that $\dim L = \dim K - 1$. Since $\dim L < \dim M - 1$ we have $\dim K < \dim M$, i.e. $J \subseteq I^M_{\{1\}}$. □

We now sketch a proof of Proposition 2.7.

Proof of Proposition 2.7. Note that by 2.6 we have $c_d(I,M) = c_d(J,M)$, so all we need to prove is that $c_i(I,M) = c_i(J,M)$ for $i = 0,\dots,d-1$. We use induction on $d = \dim M$.

If $d = 0,1$, the conclusion follows immediately from 2.6. Replacing M by I^kM for a suitable k, we may assume that $\operatorname{depth}_I(M) > 0$. Indeed, I is a reduction of (J,M), so there exists a positive integer n such that $IJ^nM = J^{n+1}M$. Then for $j \gg 0$,

$$\lambda\Big(\frac{J^jI^kM}{\mathfrak{m}^{i+1}J^jI^kM + J^{j+1}I^kM}\Big) = \lambda\Big(\frac{J^{j+k}M}{\mathfrak{m}^{i+1}J^{j+k}M + J^{j+1+k}M}\Big), \tag{4.5.3}$$

which implies that $c_i(J,M) = c_i(J,I^kM)$ for $i = 0,1,\dots,d-1$.

Choose $x \in I$ a nonzerodivisor on M which is a superficial element for (I,M) and (J,M). By Proposition 2.11, we have $c_i(I,M) = c_i(\overline{I},\overline{M})$ and $c_i(J,M) = c_i(\overline{J},\overline{M})$ for $i = 0,\dots,d-1$, where for an A module L we denote $\overline{L} = L/xL$. The induction hypothesis implies that $c_i(I,M) = c_i(J,M)$ for $i = 0,\dots,d-1$. □

4.6. Example. Let $A = k[x,y,z]$ be the ring of polynomials in three variables over the field k, and let $\mathfrak{m} = (x,y,z)$ be the maximal homogeneous ideal. As in the local case, one can define the generalized Hilbert coefficients and the first coefficient ideal associated to an ideal.

Let $I = (x^5,y^3,xyz^2)$ and let $J = (x^5,y^3,xyz^2,x^4y^2)$. Note that both ideals have height 2 and analytic spread 3. A computation with Macaulay 2 [11] shows that $j_0(I) = j_0(J) = 30$, $j_1(I) = j_1(J) = (8,-32)$, $j_2(I) = (0,-1,5)$, $j_2(J) = (0,-1,3)$.

In fact, using the method described in [7, Proposition 3.2], one can show that $J = I_{\{1\}}$, hence the equality of the first two generalized Hilbert coefficients.

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