ASYMPTOTIC PRIMES OF S_2-FILTRATIONS

CĂTĂLIN CIUPERCĂ

Abstract. Let A be a noetherian ring with total ring of fractions $Q(A)$ and $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property of Serre. Under mild conditions on the ring A, we study the behavior of the sets of associated prime ideals $\mathrm{Ass}(A/I_n \cap A)$ for $n \geq 1$. In particular, we consider the case when S is the S_2-ification of the extended Rees algebra of an ideal I. As applications, we obtain several results regarding the asymptotic behavior of $\mathrm{Ass}(A/I^n)$ for certain ideals of analytic deviation one. We also prove several consequences about the symbolic powers of a prime ideal.

1. Introduction

Let A be a noetherian ring and I an ideal in A. Answering a question of Ratliff regarding the behavior of $\mathrm{Ass}(A/I^n)$ for n large, Brodmann proved that the sequence of sets $\mathrm{Ass}(A/I^n)$ stabilizes for n large enough. In the same vein, if we consider the filtration of integral closures $\{\overline{I^n}\}_{n\geq 1}$, results of Ratliff show that the sets $\mathrm{Ass}(A/\overline{I^n})$ stabilize as well. Moreover, $\mathrm{Ass}(A/\overline{I^n}) \subseteq \mathrm{Ass}(A/\overline{I^{n+1}})$ for all n, a property which is not always satisfied by the sets $\mathrm{Ass}(A/I^n)$. If $A^*(I)$ and $\overline{A}^*(I)$ denote the stabilizing sets of $\mathrm{Ass}(A/I^n)$ and $\mathrm{Ass}(A/\overline{I^n})$, respectively, it is also known that $\overline{A}^*(I) \subseteq A^*(I)$. We refer to the monograph of McAdam [15] for a detailed exposition of these properties.

The asymptotic behavior of these sets of associated primes is best studied by considering the extended Rees algebra $\mathcal{R} = A[It, t^{-1}]$. If $\overline{\mathcal{R}}$ denotes the integral closure of $\mathcal{R}$ in its total quotient ring, we have $\overline{\mathcal{R}} \subseteq Q(A)[t, t^{-1}]$, where $Q(A)$ is the total ring of fractions of A, and $\overline{\mathcal{R}} \cap A[t, t^{-1}] = \oplus_{n\in\mathbb{Z}} \overline{I^n} t^n$. The properties (R_1) and (S_2) of Serre derived from the integral closedness of the ring $\overline{\mathcal{R}}$ can then be exploited to deduce the nice behavior of the sets $\mathrm{Ass}(A/\overline{I^n})$. In this paper we consider finite graded extensions $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ of the extended

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Rees algebra $A[It, t^{-1}]$ inside $Q(A)[t, t^{-1}]$ that only satisfy the (S_2) property of Serre. Under mild conditions on the ring A, we show that the filtration of ideals $\{I_n \cap A\}_{n\geq 1}$, which we refer to as an S_2-filtration, has similar asymptotic properties with respect to its associated prime ideals. More precisely, the sets $\operatorname{Ass}(A/I_n \cap A)$ form an increasing sequence that eventually stabilizes and $\bigcup_{n\geq 1} \operatorname{Ass}(A/I_n \cap A) = \bigcup_{n\geq 1} \operatorname{Ass}(A/\overline{I^n})$ (Corollary 2.8). It is not the purpose of this note to recover the already known results about the asymptotic behavior of $\operatorname{Ass}(A/\overline{I^n})$. In fact, many of our arguments assume that the ring A is a universally catenary domain, a constraint not imposed in the work of Ratliff regarding the behavior of $\operatorname{Ass}(A/\overline{I^n})$. Our main goal is to apply these properties in a minimal finite birational extension of $\mathcal{R}$ that satisfies the (S_2) property, the so-called S_2-ification of $\mathcal{R}$ (Corollary 2.15). With respect to the arguments used in our proofs, since we are only assuming the (S_2) property, we can no longer employ the use of certain discrete valuation rings which, in the case of the filtration $\{\overline{I^n}\}_{n\geq 1}$, were obtained by localizing $\overline{\mathcal{R}}$ at the minimal prime ideals of $t^{-1}\overline{\mathcal{R}}$ (cf. [15, 3.1–3.3]). Additionally, some other mild assumptions on the ring need to be made in order to ensure the existence of an S_2-ification.

Another motivating result for our study is a characterization due to Ratliff [17, Theorem 4.3] of the Cohen-Macaulay rings in terms of the sets $A^*(I)$ and $\overline{A}^*(I)$. More precisely, if A is a locally formally equidimensional ring, then A is Cohen-Macaulay if and only if $A^*(I) = \overline{A}^*(I)$ for all ideals I of the principal class. By using our results regarding $A^*(I)$ when $\mathcal{R}$ satisfies (S_2), we conclude that A is Cohen-Macaulay if and only if $A[It, t^{-1}]$ satisfies (S_2) for every ideal I of the principal class (Proposition 2.17).

All these results are obtained in Section 2 where we develop the main ideas in the general context of graded noetherian algebras $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property. In Section 3 we obtain several consequences regarding the symbolic powers of a prime ideal. In particular, if $\mathfrak{p}$ is a prime ideal in a universally catenary domain A such that the associated graded ring $G_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ satisfies (S_1), we show that $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$ if and only if the n-th symbolic power $\mathfrak{p}^{(n)}$ coincides with the degree n component in A of the S_2-ification of $\mathcal{R}$ for all $n \geq 1$ (Theorem 3.4). This improves a similar characterization of Huckaba who, under the global assumption that $G_{\mathfrak{p}}(A)$ satisfies (S_1), proved that $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$ if and only if $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for all n (Corollary 3.5). We note that the equimultiple ideals are important cases of ideals that satisfy the condition $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$. We also provide an example of an equimultiple ideal $\mathfrak{p}$ in a Cohen-Macaulay domain such that $G_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})$ satisfies (S_1) but $G_\mathfrak{p}(A)$ does not, and compute several symbolic powers of $\mathfrak{p}$ by using the S_2-ification of its extended Rees algebra. In Section 4 we recover a result of Brodmann showing that, for an almost complete intersection ideal I, the sets $\operatorname{Ass}(A/I^n)$ form an increasing sequence and the height of any prime in $A^*(I)$ is at most $\operatorname{ht} I + 1$ (Proposition 4.3). Similar properties are also obtained for certain ideals I of analytic spread $\ell(I) = \operatorname{ht} I + 1$ (Proposition 4.4). In both situations, the main observation is that the extended Rees algebras of the ideals involved satisfy the (S_2) property, which allows us to apply Corollary 2.10.

2. S_2-filtrations

Throughout this paper all the rings are commutative with identity. A noetherian local ring $(A, \mathfrak{m})$ is said to be formally equidimensional if all the minimal primes of the completion $\widehat{A}$ have the same dimension. A noetherian ring is called locally formally equidimensional if all of its localizations are formally equidimensional. It is known that a locally formally equidimensional ring is universally catenary and, if A is a noetherian domain, A is locally formally equidimensional if and only if A is universally catenary. We refer to the Appendix of [12] for a brief account of these properties and for terminology and concepts not otherwise explained in this paper.

2.1. The (S_i) property. Let M be a finitely generated module over a noetherian ring A. We say that M satisfies Serre's (S_i) property if for every $\mathfrak{p} \in \operatorname{Spec}(A)$ we have

$$\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \dim M_\mathfrak{p}\}.$$

We caution that a slightly different definition is sometimes used in the literature by requiring the stronger condition $\operatorname{depth} M_\mathfrak{p} \geq \min\{i, \operatorname{ht} \mathfrak{p}\}$. The conditions are clearly equivalent if $\operatorname{Ann}_A M = (0)$.

We say that the ring A satisfies the (S_2) property if A satisfies the (S_2) property as an A-module. Equivalently, A has no embedded prime ideals (i.e. A satisfies the (S_1) property) and, for every regular element $x \in A$, the ring A/xA has no embedded prime ideals.

2.2. If $A \hookrightarrow B$ is a finite extension of noetherian rings and B satisfies the (S_2) property as an A-module, then B satisfies the (S_2) property as a ring.

Moreover, if the extension satisfies the condition

$$\operatorname{ht} \mathfrak{q}_1 = \operatorname{ht} \mathfrak{q}_2 \text{ for every } \mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec} B \text{ with } \mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A,$$

then the converse also holds, i.e., B satisfies (S_2) as a ring if and only if B satisfies (S_2) as an A-module ([14, 5.7.11]). In particular, if A is a universally catenary domain and $A \hookrightarrow B$ is a finite birational extension, since $\operatorname{ht} \mathfrak{q} = \operatorname{ht}(\mathfrak{q} \cap A)$ for every $\mathfrak{q} \in \operatorname{Spec} B$ ([12, 4.8.6], it follows that B satisfies (S_2) as a ring if and only if B satisfies (S_2) as an A-module.

In the above context, when we say that B satisfies the (S_2) property we mean that B satisfies the (S_2) property as a ring.

Definition 2.3. Let A be a noetherian ring with total ring of fractions $Q(A)$. If $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ is a noetherian graded ring with $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property, we say that the family of ideals $\mathcal{F} = \{I_n \cap A\}_{n \geq 1}$ is an S_2-filtration of ideals in A.

The following Proposition contains results that are well-known in the case when S is the integral closure of an extended Rees algebra $A[It, t^{-1}]$. Our statements are adaptations for the more general context of certain graded algebras that only satisfy the (S_2) property. Most arguments used in its proof are also variations of known techniques.

Proposition 2.4. *Let A be a noetherian ring and $S = \oplus_{n \in \mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property. Let $Q_1, Q_2, \ldots, Q_m$ be the minimal prime ideals of the S-module $S/t^{-1}S$. The following are true:*

(a) $I_n \cap A = \bigcap_{i=1}^{m} t^{-n} S_{Q_i} \cap A$ *for all* $n \geq 1$*;*
(b) $\bigcup_{n \geq 1} \operatorname{Ass}(A/I_n \cap A) \subseteq \{Q_1 \cap A, \ldots, Q_m \cap A\}$*;*
(c) *If* $S \subseteq A[t, t^{-1}]$*, then*

$$\bigcup_{n \geq 1} \operatorname{Ass}(A/I_n) = \{Q_1 \cap A, \ldots, Q_m \cap A\};$$

(d) *If A is a universally catenary domain, S is a finitely generated A-algebra, and S is finite over $S \cap A[t,t^{-1}]$, then*

$$\bigcup_{n\geq 1} \operatorname{Ass}(A/I_n \cap A) = \{Q_1 \cap A, \dots, Q_m \cap A\}.$$

Proof. (a) Since S satisfies the (S_2) property, for each $n \geq 1$ we have $\operatorname{Ass}(S/t^{-n}S) = \operatorname{Min}(S/t^{-1}S) = \{Q_1, \dots, Q_m\}$, so $t^{-n}S = \bigcap_{i=1}^{m} t^{-n}S_{Q_i} \cap S$. As $I_n = t^{-n}S \cap Q(A)$, the conclusion follows.

(b) The ideal $t^{-n}S_{Q_i}$ is a $Q_iS_{Q_i}$-primary ideal in S_{Q_i}, so $t^{-n}S_{Q_i} \cap A$ is a primary ideal in A. Then, for each n,

$$I_n \cap A = \bigcap_{i=1}^{m} (t^{-n}S_{Q_i} \cap A)$$

is a (possibly redundant) primary decomposition of $I_n \cap A$, and therefore

$$\bigcup_{n\geq 1} \operatorname{Ass}(A/I_n \cap A) \subseteq \{Q_1S_{Q_1} \cap A, \dots, Q_mS_{Q_m} \cap A\} = \{Q_1 \cap A, \dots, Q_m \cap A\}.$$

(c) Let Q be a minimal prime ideal of $S/t^{-1}S$. Write $Q = (t^{-1}S :_S at^s)$ for some homogeneous element $at^s \in S$ ($a \in I_s$). Then

$$Q \cap A = (t^{-1}S :_A at^s) = (I_{s+1} :_A a),$$

which shows that $Q \cap A \in \operatorname{Ass}(A/I_{s+1})$.

(d) Let Q be a minimal prime ideal of $S/t^{-1}S$. Denote $S' = S \cap A[t,t^{-1}]$ and $Q' = Q \cap S'$. Since $A \subseteq S' \subseteq S$, S is a finitely generated A-algebra, and S is a module-finite extension of S', by a result of Artin and Tate [1, Theorem 1], the ring S' is a finitely generated A-algebra, and therefore a universally catenary domain. Then, by [12, Proposition 4.8.6], we have $\operatorname{ht} Q' = \operatorname{ht} Q = 1$, so Q' is a minimal prime of $S'/t^{-1}S'$. As in part (c), we can now write $Q' = (t^{-1}S' :_{S'} at^s)$ for some homogeneous element $at^s \in S'$ ($a \in I_s \cap A$). Then

$$Q \cap A = Q' \cap A = (t^{-1}S' :_A at^s) = ((I_{s+1} \cap A) :_A a),$$

which shows that $Q \cap A \in \operatorname{Ass}(A/I_{s+1} \cap A)$. ∎

Remark 2.5. A similar argument to the one used above in part (c) shows that for every ideal I in a noetherian ring A we have

$$\bigcup_{n\geq 1} \operatorname{Ass}(A/I^n) \supseteq \{P_1 \cap A, \dots, P_r \cap A\},$$

where $\{P_1, \dots, P_r\} = \operatorname{Min}(A[It, t^{-1}]/(t^{-1}))$.

In order to prove the main results of this section we need the following lemma.

Lemma 2.6. *Let* $(A, \mathfrak{m})$ *be a formally equidimensional local ring and let* $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ *be a noetherian graded ring such that* $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ *and* $\operatorname{ht}(I_1 \cap A) \geq 1$*. Let* $R = A[(I_1 \cap A)t, t^{-1}] \subseteq S$*. Assume that* S *is integral over* R *and that either*

(i) *A is a domain; or*
(ii) *$S \subseteq A[t, t^{-1}]$.*

Then

$$\operatorname{ht}((I_1 \cap A)t, t^{-1})S \geq 2.$$

Proof. By contradiction, assume that there exists a prime Q containing $((I_1 \cap A)t, t^{-1})S$ such that $\operatorname{ht} Q = 1$. Let $P = Q \cap R$.

If the condition (i) is satisfied, then R is a universally catenary domain and $R \hookrightarrow S$ is an integral extension. By using again [12, 4.8.6], we obtain $\operatorname{ht} P = \operatorname{ht} Q = 1$, so P is a prime in the extended Rees algebra R that is minimal over $t^{-1}R$. As A is formally equidimensional, by [12, 5.4.8] we have $\dim R/P = \dim A$. On the other hand, since $R/((I_1 \cap A)t, t^{-1})R \cong A/I_1 \cap A$, we have $\dim R/P \leq \dim A/I_1 \cap A$. This implies $I_1 \cap A = (0)$, contradiction.

We now assume that the condition (ii) is satisfied. We first show that every minimal prime ideal of S is of the form $\mathfrak{p}A[t, t^{-1}] \cap S$ with $\mathfrak{p}$ a minimal prime of A. To see this, for every ideal J in A, denote $J_S = JA[t, t^{-1}] \cap S$. One can immediately check that if $\mathfrak{p}$ is a prime ideal in A, then $\mathfrak{p}_S$ is a prime ideal in S, and if $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in A, then $\mathfrak{q}_S$ is a $\mathfrak{p}_S$-primary ideal in S. Moreover, $J_S \cap A = J$ for every ideal J in A. Consequently, if $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$ is an irredundant primary decomposition of the zero ideal in A, then $(0) = (\mathfrak{q}_1)_S \cap \dots \cap (\mathfrak{q}_r)_S$ is an irredundant primary decomposition of the zero ideal in S. This shows that the ideals $\{\mathfrak{p}_S \mid \mathfrak{p} \in \operatorname{Min}(A)\}$ are all the minimal prime ideals of S. Similarly, the ideals $\{\mathfrak{p}_R \mid \mathfrak{p} \in \operatorname{Min}(A)\}$ are all the minimal prime ideals of R.

Now let $\mathfrak{p}_S = \mathfrak{p}A[t, t^{-1}] \cap S$ be a minimal prime ideal of S such that $Q \supseteq \mathfrak{p}_S$. Then $\mathfrak{p}_R = \mathfrak{p}_S \cap R$ is a minimal prime ideal of R that is contained in P. Note that

$$R/\mathfrak{p}_R = R/(\mathfrak{p}A[t, t^{-1}] \cap R) \cong (A/\mathfrak{p})\Big[\frac{I_1 \cap A + \mathfrak{p}}{\mathfrak{p}}t, t^{-1}\Big]$$

is the extended Rees algebra of the ring $A/\mathfrak{p}$ with respect to the image of the ideal $I_1 \cap A$. In particular,

$$\dim R/\mathfrak{p}_R = \dim A/\mathfrak{p} + 1 = \dim A + 1.$$

As $A/\mathfrak{p}$ is a formally equidimensional local domain and hence universally catenary, $R/\mathfrak{p}_R$ is also universally catenary. Then, if $\mathfrak{M} = ((I_1 \cap A)t, \mathfrak{m}, t^{-1})R$ denotes the maximal homogeneous ideal of R, we have

$$\text{(2.6.1)}\quad \dim A+1 = \dim R/\mathfrak{p}_R = \operatorname{ht}(\mathfrak{M}/\mathfrak{p}_R) = \operatorname{ht}(\mathfrak{M}/P)+\operatorname{ht}(P/\mathfrak{p}_R) = \dim R/P+\operatorname{ht}(P/\mathfrak{p}_R).$$

Moreover, since $R/\mathfrak{p}_R \hookrightarrow S/\mathfrak{p}_S$ is an integral extension, by [12, 4.8.6] we have $\operatorname{ht}(P/\mathfrak{p}_R) = \operatorname{ht}(Q/\mathfrak{p}_S) = 1$. Therefore (2.6.1) implies that $\dim R/P = \dim A$. However, since $R/((I_1 \cap A)t, t^{-1})R \cong A/I_1 \cap A$, we have $\dim R/P \leq \dim A/I_1 \cap A$, contradicting $\operatorname{ht}(I_1 \cap A) \geq 1$. □

Theorem 2.7. *Let A be a locally formally equidimensional ring and let $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ be a noetherian graded ring such that $A[t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property. Let $R = A[(I_1 \cap A)t, t^{-1}] \subseteq S$. Assume that S is integral over R and that either*

(i) *A is a domain; or*
(ii) *$S \subseteq A[t, t^{-1}]$ and $\operatorname{ht} I_1 \geq 1$.*

Then

$$\operatorname{Ass}(A/I_n \cap A) \subseteq \operatorname{Ass}(A/I_{n+1} \cap A) \text{ for all } n \geq 1.$$

Proof. First note that under the assumption (i) we may also assume that $\operatorname{ht}(I_1 \cap A) \geq 1$, i.e. $I_1 \cap A \neq (0)$, for otherwise we must have $I_n = 0$ for all $n \geq 1$.

Let $Q \in \operatorname{Ass}(A/I_n \cap A)$. After localizing at Q we may assume that A is a formally equidimensional local ring with maximal ideal Q. Moreover, by passing to the faithfully flat extension $A[X]_{QA[X]}$, we may also assume that the residue field of A is infinite. Write $Q =$

$((I_n \cap A) :_A b) = (t^{-n}S :_S b) \cap A$ for some $b \in A$. Since S satisfies the (S_2) property, by Lemma 2.6 and homogeneous prime avoidance, there exists $at \in (I_1 \cap A)t$ that does not belong to any of the prime ideals in $\mathrm{Ass}(S/t^{-n}S) = \mathrm{Min}(S/t^{-1}S)$. As $\mathrm{Ass}(S/(t^{-n}S :_S b)) \subseteq \mathrm{Ass}(S/t^{-n}S)$, the element at is a non-zero-divisor on $S/(t^{-n}S :_S b)$, so we have $Q = (t^{-n}S :_S abt) \cap A$. Then $Q = ((I_{n+1} \cap A) :_A ab)$, which shows that $Q \in \mathrm{Ass}(A/I_{n+1} \cap A)$. □

Corollary 2.8. *Let A be a locally formally equidimensional domain, I an ideal in A, and $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[It, t^{-1}] \subseteq S \subseteq Q(A)[t, t^{-1}]$ and S satisfies the (S_2) property. Assume that S is a finite extension of $A[It, t^{-1}]$. Then the following hold:*

(a) $\mathrm{Ass}(A/I_n \cap A) \subseteq \mathrm{Ass}(A/I_{n+1} \cap A)$ *for all* $n \geq 1$*;*
(b) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I_n \cap A) = \{Q \cap A \mid Q \in \mathrm{Min}(S/t^{-1}S)\}$*;*
(c) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I_n \cap A) = \bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n})$.

Proof. Parts (a) and (b) are immediate consequences of Theorem 2.7 and Proposition 2.4 (d).

For part (c), let $T = \overline{A[It, t^{-1}]}$ be the integral closure of $A[It, t^{-1}]$ in its quotient field. Note that $S \subseteq T = \oplus_{n\in\mathbb{Z}} \overline{I^n \overline{A}} t^n \subseteq Q(A)[t, t^{-1}]$ ([12, 5.2.4]) and $T \cap A[t, t^{-1}] = \oplus_{n\in\mathbb{Z}} \overline{I^n} t^n$. From the classical result regarding the asymptotic primes of the filtration $\{\overline{I^n}\}_{n\geq 1}$ we know that $\bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n}) = \{Q \cap A \mid Q \in \mathrm{Min}(T/t^{-1}T)\}$ (see, for example, [15, Proposition 3.18]). Because of (b), if we show that the contractions to S of all the primes in $\mathrm{Min}_T(T/t^{-1}T)$ give us all the primes in $\mathrm{Min}_S(S/t^{-1}S)$, the proof is finished. For this, note that S is a universally catenary domain and $S \subseteq T$ is an integral extension. By [12, 4.8.6], for every prime Q minimal over $t^{-1}T$ we have $\mathrm{ht}(Q \cap S) = \mathrm{ht}\, Q = 1$, so $Q \cap S$ is minimal over $t^{-1}S$. Moreover, if P is a prime minimal over $t^{-1}S$, by the lying over property for integral extensions there exists a prime Q containing $t^{-1}T$ such that $P = Q \cap S$. We also have $\mathrm{ht}\, Q \leq \mathrm{ht}\, P$, so Q must be minimal over $t^{-1}T$. □

When A is not necessarily a domain, but $S \subseteq A[t, t^{-1}]$, we obtain similar conclusions in parts (a) and (b), and only one inclusion in part (c).

Corollary 2.9. *Let A be a locally formally equidimensional ring, I an ideal in A, and $S = \oplus_{n\in\mathbb{Z}} I_n t^n$ a noetherian graded ring such that $A[It, t^{-1}] \subseteq S \subseteq A[t, t^{-1}]$, $\mathrm{ht}\, I_1 \geq 1$, and*

S satisfies the (S_2) property. Assume that S is a finite extension of $A[It, t^{-1}]$. Then the following hold:

(a) $\mathrm{Ass}(A/I_n) \subseteq \mathrm{Ass}(A/I_{n+1})$ *for all* $n \geq 1$*;*

(b) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I_n) = \{Q \cap A \mid Q \in \mathrm{Min}(S/t^{-1}S)\}$*;*

(c) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I_n) \subseteq \bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n})$.

Proof. Parts (a) and (b) follow from Theorem 2.7 (ii) and Proposition 2.4 (c). Moreover, by following the line of proof of part (c) of Corollary 2.8 without assuming that A is domain, it is still true that every minimal prime of $S/t^{-1}S$ is contracted from a minimal prime of $T/t^{-1}T$, and therefore

$$\bigcup_{n\geq 1} \mathrm{Ass}(A/I_n \cap A) \subseteq \bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n}).$$

□

In particular, when S is an extended Rees algebra, we have the following.

Corollary 2.10. *Let A be a locally formally equidimensional ring and I an ideal in A with* $\mathrm{ht}\, I \geq 1$*. Assume that the extended Rees algebra $A[It, t^{-1}]$ satisfies the (S_2) property. The following hold:*

(a) $\mathrm{Ass}(A/I^n) \subseteq \mathrm{Ass}(A/I^{n+1})$ *for every* $n \geq 1$*;*

(b) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I^n) = \{Q \cap A \mid Q \in \mathrm{Min}(A[It, t^{-1}]/(t^{-1}))$;

(c) $\bigcup_{n\geq 1} \mathrm{Ass}(A/I^n) = \bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n})$.

Proof. Parts (a) and (b) are direct consequences of Corollary 2.9. From the classical result of Ratliff we have $\bigcup_{n\geq 1} \mathrm{Ass}(A/\overline{I^n}) = \overline{A}^*(I) \subseteq A^*(I)$ and, by part (a), $A^*(I) = \bigcup_{n\geq 1} \mathrm{Ass}(A/I^n)$. This shows the inclusion "$\supseteq$" of part (c). The other inclusion follows from Corollary 2.9(c).

□

2.11. The S_2-ification of a noetherian domain. For a noetherian domain A with quotient field $Q(A)$, an S_2-ification of A is a birational extension $S_2(A)$ of A that is minimal among the finite birational extensions of A that satisfy the (S_2) property as A-modules. If A has an S_2-ification, then it is unique. More precisely, A has an S_2-ification $S_2(A)$ if and only if $C := \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} A_{\mathfrak{p}}$ is a finite extension of A, in which case C is the S_2-ification of A.

If A is a universally catenary domain such that the extension $A \subseteq \overline{A}$ is finite (which holds, for instance, when A is an analytically unramified local domain), then A has an S_2-ification [14, 5.11.2]. Also, if A has a canonical module ω, then $\mathrm{Hom}_A(\omega, \omega)$ is the S_2-ification of A [16, Theorem 1.3]. We refer to [6, 16, 19] for detailed accounts on S_2-ification.

2.12. The S_2-ification of $A[It, t^{-1}]$. Let A be a noetherian domain and I an ideal in A. Assume that $\mathcal{R} = A[It, t^{-1}]$ has an S_2-ification $S_2(\mathcal{R})$. (The S_2-ification $S_2(\mathcal{R})$ exists for a large class of rings; it does, for example, when A is a universally catenary analytically unramified local domain.) As explained in [4, 2.3], $S_2(\mathcal{R})$ is a graded subring of $Q(A)[t, t^{-1}]$, so that we can write $S_2(\mathcal{R}) = \oplus_{n \in \mathbb{Z}} I_n t^n \subseteq Q(A)[t, t^{-1}]$. The A-modules I_n are contained in the S_2-ification $S_2(A)$ of A [4, Lemma 2.4]. In particular, if A satisfies (S_2), then I_n is an ideal in A for all $n \geq 1$.

Definition 2.13. For every ideal I in a ring A such that the conditions in (2.12) are satisfied we define $S_2(I) := I_1 \cap A$, the S_2-closure of I.

Remark 2.14. Since the n-th Veronese subring of $S_2(A[It, t^{-1}])$ is the S_2-ification of the n-th Veronese subring of $A[It, t^{-1}]$, we have $S_2(I^n) = I_n \cap A$ for all $n \geq 1$.

From Corollary 2.8 we have the following consequence regarding the asymptotic primes of the S_2-filtration $S_2(I^n)$.

Corollary 2.15. *Let A be a universally catenary domain and I an ideal of A. Assume that the extended Rees algebra $\mathcal{R} = A[It, t^{-1}]$ has an S_2-ification . Then:*

(a) $\mathrm{Ass}(A/S_2(I^n)) \subseteq \mathrm{Ass}(A/S_2(I^{n+1}))$ *for every* $n \geq 1$*;*

(b) $\bigcup_{n \geq 1} \mathrm{Ass}(A/S_2(I^n)) = \bigcup_{n \geq 1} \mathrm{Ass}(A/\overline{I^n})$.

Remark 2.16. Let A be a noetherian ring that satisfies the (S_2) property and I an ideal. Consider the statements: (a) $A[It, t^{-1}]$ satisfies the (S_2) property; (b) $G_I(A)$ satisfies the (S_1) property ; and (c) $A[It]$ satisfies the (S_2) property. The following are true:

(1) (a) $\iff$ (b), because $G_I(A) \cong A[It, t^{-1}]/(t^{-1})$;

(2) If $\mathrm{ht}\, I \geq 1$, then (c) $\implies$ (b) ([3, Theorem 1.5]).

(3) If $I_{\mathfrak{p}}$ is principal for every $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht}\mathfrak{p} = 1$, then (b) $\implies$ (c) ([3, Theorem 1.5]); in particular, if $\operatorname{ht} I \geq 2$, then (b) $\iff$ (c).

The equivalence (a) $\iff$ (e) in the following Proposition was originally established by Ratliff in [17]. In view of our study of the asymptotic primes of S_2-filtrations, we are able to add several equivalent statements. In particular, the equivalence (a) $\iff$ (c) gives a characterization of the Cohen-Macaulay rings in terms of the (S_2) property of their extended Rees algebras with respect to ideals of the principal class. Recall that an ideal I is said to be of the principal class if I can be generated by $\operatorname{ht} I$ elements.

Proposition 2.17. *Let A be a locally formally equidimensional ring. The following are equivalent:*

(a) *A is Cohen-Macaulay;*

(b) *$\mathcal{R} = A[It, t^{-1}]$ is Cohen-Macaulay for every ideal I of the principal class;*

(c) *$\mathcal{R} = A[It, t^{-1}]$ satisfies the (S_2) property for every ideal I of the principal class;*

(d) *$\bigcup_{n\geq 1} \operatorname{Ass}(A/I^n) = \bigcup_{n\geq 1} \operatorname{Ass}(A/\overline{I^n})$ for every ideal I of the principal class;*

(e) *$A^*(I) = \overline{A}^*(I)$ for every ideal I of the principal class.*

Proof. The implication (a) $\implies$ (b) holds because in a Cohen-Macaulay ring an ideal of the principal class is generated by a regular sequence. By Corollary 2.10, we also obtain (c) $\implies$ (d). For (d) $\implies$ (e) note that $A^*(I) \subseteq \bigcup_{n\geq 1} \operatorname{Ass}(A/I^n) = \bigcup_{n\geq 1} \operatorname{Ass}(A/\overline{I^n}) = \overline{A}^*(I) \subseteq A^*(I)$, hence $A^*(I) = \overline{A}^*(I)$. For the implication (e) $\implies$ (a) we refer to the original paper of Ratliff [17, Theorem 4.3] or [15, Theorem 8.12]. $\square$

3. Symbolic Powers of Ideals

In this section we obtain several consequences regarding the behavior of the symbolic powers of a prime ideal. We begin with a well-known observation that will be used extensively.

Remark 3.1. Let A be a locally formally equidimensional ring and I an ideal in A. The following are equivalent:

(a) $\ell(IA_{\mathfrak{q}}) < \dim A_{\mathfrak{q}}$ for every prime $\mathfrak{q} \in V(I) \setminus \operatorname{Min}(A/I)$;

(b) $\overline{A}^*(I) = \operatorname{Min}(A/I)$.

The equivalence follows from a well-known result of McAdam that states that $\mathfrak{q} \in \overline{A}^*(I)$ if and only if $\ell(IA_\mathfrak{q}) = \dim A_\mathfrak{q}$ ([15, Proposition 4.1]). The importance of the condition (a) in the context of symbolic powers of ideals has been documented in many instances (e.g. [7,8,10]). If $I = \mathfrak{p}$ is a prime in a locally formally equidimensional ring A, the condition also implies that the $\mathfrak{p}$-adic topology and the $\mathfrak{p}$-symbolic topology on A are linearly equivalent, i.e. there exists k such that $\mathfrak{p}^{(n+k)} \subseteq \mathfrak{p}^n$ for all $n \geq 1$ ([18, Corollary 1]). We also note that if I is an equimultiple ideal in a local ring, then I satisfies the condition (a).

Recall that for an ideal I in a noetherian ring A, the unmixed part I^{unm} is the intersection of the primary components of I that correspond to the minimal prime ideals over I. For $n \geq 1$ and a prime ideal $\mathfrak{p}$, the ideal $(\mathfrak{p}^n)^{\text{unm}}$ is referred to as the n-th symbolic power of $\mathfrak{p}$ and is typically denoted by $\mathfrak{p}^{(n)}$.

We now state a non-local version of a result we proved in [5].

Proposition 3.2. *Let A be a locally formally equidimensional domain and I an ideal in A such that $\ell(IA_\mathfrak{q}) < \dim A_\mathfrak{q}$ for every prime $\mathfrak{q} \in V(I) \setminus \operatorname{Min}(A/I)$. Assume that the extended Rees algebra $\mathcal{R} = A[It, t^{-1}]$ has an S_2-ification $S_2(\mathcal{R}) = \bigoplus_{n \in \mathbb{Z}} I_n t^n$. Then, for $n \geq 1$,*

$$(I^n)^{\text{unm}} \subseteq S_2(I^n).$$

Proof. Since $\ell(IA_\mathfrak{q}) = \ell(I^n A_\mathfrak{q})$ and the n-th Veronese subring of $S_2(\mathcal{R})$ is an S_2-ification of the n-th Veronese subring of $\mathcal{R}$, it is enough to prove the conclusion for $n = 1$. Furthermore, it is enough to prove that for every prime $\mathfrak{p}$ that contains I we have $(I_\mathfrak{p})^{\text{unm}} \subseteq (I_1)_\mathfrak{p}$. We may also assume that $\mathfrak{p} \notin \operatorname{Min}(A/I)$, for otherwise $(I_\mathfrak{p})^{\text{unm}} = I_\mathfrak{p}$ and the inclusion is clear. We now note that if $\mathfrak{p} \in \operatorname{Spec}(R)$, then $\bigoplus_{n \in \mathbb{Z}} (I_n)_\mathfrak{p} t^n$ is an S_2-ification of $A_\mathfrak{p}[I_\mathfrak{p} t, t^{-1}]$. Hence, after localizing at $\mathfrak{p}$, we may assume that $(A, \mathfrak{m})$ is a formally equidimensional ring and I is an ideal in A such that $\ell(IA_\mathfrak{q}) < \dim A_\mathfrak{q}$ for every prime $\mathfrak{q} \in V(I) \setminus \operatorname{Min}(A/I)$; we need to show that $I^{\text{unm}} \subseteq I_1 \cap A$. This was proved in [5, Lemma 3.10]. Even though the statement of [5, Lemma 3.10] assumes that I is equimultiple, the proof given there works for any ideal I such that $\ell(IA_\mathfrak{q}) < \dim A_\mathfrak{q}$ for every prime $\mathfrak{q} \in V(I) \setminus \operatorname{Min}(A/I)$. Also, in [5], the additional conditions on the ring were imposed just to guarantee the existence of the S_2-ification of the extended Rees algebra. □

Remark 3.3. In the previous proposition, the requirement that A be a domain was imposed just to ensure that we are in the setup used in the description of the S_2-ification process in (2.12) where we construct the S_2-ification $S_2(\mathcal{R})$ inside the quotient field $Q(\mathcal{R})$. If the extended Rees algebra $\mathcal{R}$ already satisfies the (S_2) property, a particular case in which Proposition 3.2 will be applied subsequently, there is no need to assume that A is a domain.

We now consider a prime ideal $\mathfrak{p}$ that satisfies the equivalent conditions in Remark 3.1. Under mild conditions on the ring A, the next result shows that if the associated graded ring $G_{\mathfrak{p}A\mathfrak{p}}(A_\mathfrak{p})$ satisfies (S_1), then the filtration of symbolic powers coincides with the S_2-filtration $S_2(\mathfrak{p}^n)$. As discussed in more detail at the end of the proof of this result, previous results in the literature established that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for all n if $G_\mathfrak{p}(A)$ satisfies (S_1), and $\mathfrak{p}^{(n)} = \overline{\mathfrak{p}^n}$ for all n if $G_{\mathfrak{p}A\mathfrak{p}}(A_\mathfrak{p})$ is reduced.

Theorem 3.4. *Let A be a locally formally equidimensional domain and $\mathfrak{p}$ a prime ideal in A such that the extended Rees algebra $A[\mathfrak{p}t, t^{-1}]$ has an S_2-ification. Assume that $G_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})$ satisfies (S_1). The following are equivalent:*

(a) $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$*;*

(b) $\mathfrak{p}^{(n)} = S_2(\mathfrak{p}^n)$ *for all* $n \geq 1$*;*

(c) $\mathfrak{p}^{(n)} = S_2(\mathfrak{p}^n)$ *for infinitely many* n*.*

Proof. We first prove $(a) \Rightarrow (b)$. By Remark 3.1, we can apply Proposition 3.2 and obtain that $\mathfrak{p}^{(n)} \subseteq S_2(\mathfrak{p}^n)$. To show $S_2(\mathfrak{p}^n) \subseteq \mathfrak{p}^{(n)}$, it is enough to prove that $S_2(\mathfrak{p}^n)A_\mathfrak{p} \subseteq \mathfrak{p}^n A_\mathfrak{p}$. To see this, note that since $G_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})$ satisfies (S_1), the extended Rees algebra $A_\mathfrak{p}[\mathfrak{p}A_\mathfrak{p}t, t^{-1}]$ satisfies (S_2) (Remark 2.16), so $S_2(\mathfrak{p}^n A_\mathfrak{p}) = \mathfrak{p}^n A_\mathfrak{p}$. As $S_2(\mathfrak{p}^n A_\mathfrak{p}) = S_2(\mathfrak{p}^n)A_\mathfrak{p}$, the inclusion follows.

We now prove $(c) \Rightarrow (a)$. For n large enough, by Corollary 2.15 we have

$$\overline{A}^*(\mathfrak{p}) = \operatorname{Ass}(A/S_2(\mathfrak{p}^n)) = \operatorname{Ass}(A/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}.$$

□

The implication $(a) \Rightarrow (c)$ of the next Corollary recovers a result of Huckaba [8, Theorem 2.1]. His result, even though it was stated under the assumption $G_{\mathfrak{p}}(A)$ Cohen-Macaulay, has a proof that only requires that $G_{\mathfrak{p}}(A)$ satisfy (S_1).

Corollary 3.5. *Let A be a locally formally equidimensional ring and $\mathfrak{p}$ a prime ideal in A. Assume that $G_{\mathfrak{p}}(A)$ satisfies the (S_1) property. The following are equivalent:*

(a) $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$;

(b) $A^*(\mathfrak{p}) = \{\mathfrak{p}\}$;

(c) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ *for all* $n \geq 1$;

(d) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ *for infinitely many* n.

Proof. Since $G_{\mathfrak{p}}(A)$ satisfies (S_1), the extended Rees algebra $A[\mathfrak{p}t, t^{-1}]$ satisfies (S_2). For (a) $\Longrightarrow$ (c), note that, by Proposition 3.2 and Remark 3.3, we have $\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^n$ for every $n \geq 1$, and hence equality holds. To see $(d) \Longrightarrow (b)$, note that for n large enough we have $A^*(\mathfrak{p}) = \mathrm{Ass}(A/\mathfrak{p}^n) = \mathrm{Ass}(A/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$. Finally, as $\overline{A}^*(\mathfrak{p}) \subseteq A^*(\mathfrak{p})$, the implication $(b) \Longrightarrow (a)$ also follows. □

Remark 3.6. Let A be a Cohen-Macaulay ring and $\mathfrak{p}$ a prime ideal of height $h > 0$ such that $A_{\mathfrak{p}}$ is regular and $\mathfrak{p}$ is generated by $h+1$ elements. By [3, Proposition 2.6], the Rees algebra $A[\mathfrak{p}t]$ satisfies (S_2), hence $G_{\mathfrak{p}}(A)$ satisfies (S_1) (Remark 2.16), so the conclusions of Corollary 3.5 follow. This recovers results from [2, 11]. For the local case, Huneke and Huckaba [9, Theorem 2.5] obtained the same conclusions of Corollary 3.5 under the weaker assumption that the analytic spread $\ell(\mathfrak{p}) = h + 1$.

Remark 3.7. Let A be a locally formally equidimensional domain and $\mathfrak{p}$ a prime ideal in A. Assume that $G_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is reduced. By a result of Huckaba [7, Theorem 1.4] (which extends work of Huneke [10, Theorem 2.1]), the following are equivalent:

(a) $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$;

(b) $\mathfrak{p}^{(n)} = \overline{\mathfrak{p}^n}$ for all $n \geq 1$.

Since $G_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is reduced, it satisfies the (S_1) property. Therefore, by Theorem 3.4, the above equivalent conditions imply that $S_2(\mathfrak{p}^n) = \overline{\mathfrak{p}^n}$ for all $n \geq 1$. If A is a normal domain, this shows that the extended Rees algebra $A[\mathfrak{p}t, t^{-1}]$ is regular in codimension one.

Indeed, if Q is a height one prime ideal in $A[\mathfrak{p}t, t^{-1}]$, then $A[\mathfrak{p}t, t^{-1}]_Q \cong S_2(A[\mathfrak{p}t, t^{-1}]_Q) \cong S_2(A[\mathfrak{p}t, t^{-1}])_Q$ which is a local integrally closed ring because $S_2(A[\mathfrak{p}t, t^{-1}])$ coincides with the integral closure of $A[\mathfrak{p}t, t^{-1}]$.

The following example is a modification of [7, Example 1.6]. It describes a situation where all the hypotheses of Theorem 3.4 are satisfied for an equimultiple prime ideal $\mathfrak{p}$, and hence $\mathfrak{p}^{(n)} = S_2(\mathfrak{p}^n)$ for all $n \geq 1$. On the other hand, the associated graded ring $G_{\mathfrak{p}}(A)$ does not satisfy the (S_1) property.

Example 3.8. Let $A = k[u, v, w, z, r]_{(u,v,w,z,r)}/(w^7 - u^{35}z^2 + u^{30}v, v^3 - wz)$ and the prime ideal $\mathfrak{p} = (u, v, w, r)A$. Using Macaulay2 [13] one can check that A is a three dimensional Cohen-Macaulay domain and $\operatorname{ht} \mathfrak{p} = \ell(\mathfrak{p}) = 2$, so $\mathfrak{p}$ is an equimultiple ideal. This also implies that $\overline{A}^*(\mathfrak{p}) = \{\mathfrak{p}\}$. The associated graded ring $G_{\mathfrak{p}}(A)$ has a unique minimal prime ideal $(v^*, w^*)G_{\mathfrak{p}}(A)$, where $v^*, w^* \in \mathfrak{p}/\mathfrak{p}^2 \subseteq G_{\mathfrak{p}}(A)$ are the images of v and w, respectively. Moreover, the associated prime ideals of $G_{\mathfrak{p}}(A)$ are $(v^*, w^*)G_{\mathfrak{p}}(A)$ and $(v^*, w^*, z^*)G_{\mathfrak{p}}(A)$, and hence $G_{\mathfrak{p}}(A)$ does not satisfy Serre's (S_1) property.

In the local ring $A_{\mathfrak{p}}$ we have $w = (1/z)v^3$, so $w^7 - u^{35}z^2 + u^{30}v = (1/z^7)(v^{21} - u^{35}z^9 + u^{30}vz^7)$ and therefore

$$A_{\mathfrak{p}} \cong k[u, v, z, r]_{(u,v,r)}/(v^{21} - u^{35}z^9 + u^{30}vz^7).$$

Since $A_{\mathfrak{p}}$ is a hypersurface $R/(f)$, where $R = k[u, v, z, r]_{(u,v,r)}$ and $f = v^{21} - u^{35}z^9 + u^{30}vz^7$ with initial term $f^* = v^{21}$, it follows that $G_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \cong k[u, v, z, r]_{(u,v,r)}/(v^{21})$. Note that $G_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ is not reduced, but it has a unique minimal prime ideal and no embedded associated prime ideals, so it satisfies the (S_1) property. This implies that the extended Rees algebra $A_{\mathfrak{p}}[\mathfrak{p}A_{\mathfrak{p}}t, t^{-1}]$ satisfies the (S_2) property, and therefore $S_2(\mathfrak{p}^n A_{\mathfrak{p}}) = \mathfrak{p}^n A_{\mathfrak{p}}$ for all $n \geq 1$.

On the other hand, let us note that the ideal $\mathfrak{p}A_{\mathfrak{p}}$ is not normal, i.e. not all of its powers are integrally closed. This follows from the criterion for normality of the maximal ideal in a hypersurface that is proved in [3, 2.4]. In fact, in the ring A one can check with Macaulay2 that $w^7 \in \mathfrak{p}^{28}$ and $w \notin \mathfrak{p}^2$. Then $w \in \overline{\mathfrak{p}^4} \subseteq \overline{\mathfrak{p}^2}$, hence $\overline{\mathfrak{p}^2} \neq \mathfrak{p}^2$. Using the procedure outlined in [4, Proposition 3.2] (which is valid in any affine domain), by identifying the S_2-ification of the Rees algebra $A[\mathfrak{p}t]$ with the ring of endomorphisms of the canonical ideal of $A[\mathfrak{p}t]$, after lengthy computations with Macaulay2 we were able to obtain

$$\begin{aligned} S_2(\mathfrak{p}^2) =& \mathfrak{p}^2 + (w), \\ S_2(\mathfrak{p}^3) =& \mathfrak{p}^3 + (w) \supsetneq \mathfrak{p}S_2(\mathfrak{p}^2), \\ S_2(\mathfrak{p}^4) =& \mathfrak{p}^4 + (wr, w^2, vw, uw) = \mathfrak{p}S_2(\mathfrak{p}^3) + (S_2(\mathfrak{p}^2))^2, \text{ and} \\ S_2(\mathfrak{p}^5) =& \mathfrak{p}^5 + (w^2, wr^2, vwr, v^2w, uvw, u^2w, uwr) = \mathfrak{p}S_2(\mathfrak{p}^4) + S_2(\mathfrak{p}^2)S_2(\mathfrak{p}^3). \end{aligned}$$

One can also check that $w \notin S_2(\mathfrak{p}^4)$. As noticed before, $w \in \overline{\mathfrak{p}^4} \setminus \mathfrak{p}^2$, so we have the strict inclusions

$$\mathfrak{p}^4 \subsetneq S_2(\mathfrak{p}^4) \subsetneq \overline{\mathfrak{p}^4}.$$

On the other hand, since $G_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})$ satisfies (S_1), by Theorem 3.4 we know that $\mathfrak{p}^{(n)} = S_2(\mathfrak{p}^n)$ for all $n \geq 1$. In fact, in this particular example, for $i \in \{2, 3, 4, 5\}$ we have $\mathrm{Ass}(A/\mathfrak{p}^i) = \{\mathfrak{p}, \mathfrak{m}\}$, where $\mathfrak{m} = (u, v, w, z, r)A$, so $\mathfrak{p}^{(i)} = (\mathfrak{p}^i : \mathfrak{m}^\infty)$. Compared to $S_2(\mathfrak{p}^i)$, the saturations $(\mathfrak{p}^i : \mathfrak{m}^\infty)$ are much easier to compute in Macaulay2 and we double checked that the ideals $(\mathfrak{p}^i : \mathfrak{m}^\infty)$ coincide with the ideals $S_2(\mathfrak{p}^i)$ $(2 \leq i \leq 5)$ computed above.

4. Other Applications

Proposition 4.1. *Let* $(A, \mathfrak{m})$ *be a formally equidimensional local ring and* I *an ideal of positive height and analytic spread* $\ell(I) < \dim A$*. Assume that* $A[It, t^{-1}]$ *satisfies the* (S_2) *property. Then*

$$(I^n : \mathfrak{m}^\infty) = I^n \text{ for all } n \geq 1,$$

i.e., $\mathfrak{m} \notin \bigcup_{n\geq 1} \mathrm{Ass}(A/I^n)$.

Proof. By Corollary 2.10, we have $\overline{A}^*(I) = \bigcup_{n\geq 1} \mathrm{Ass}(A/I^n)$ and $\mathfrak{m} \in \overline{A}^*(I)$ if and only if $\ell(I) = \dim A$ ([15, Proposition 4.1]). □

Remark 4.2. If the ring A is also a domain, by using very different methods we already obtained the above result in [5, Corollary 3.15].

We are also able to recover the following result of Brodmann regarding the asymptotic primes of an almost complete intersection ideal.

Proposition 4.3. [2, Proposition 3.9] *Let A be a Cohen-Macaulay ring and I an ideal of height h that can be generated by $h+1$ elements. Moreover, assume that I is generically a complete intersection. Then*

(a) $\operatorname{Ass}(A/I^n) \subseteq \operatorname{Ass}(A/I^{n+1})$ *for all* $n \geq 1$*;*
(b) $\operatorname{ht} \mathfrak{p} \leq h+1$ *for all* $\mathfrak{p} \in A^*(I)$.

Proof. Note that we may assume $h \geq 1$, for otherwise $I = (0)$ and the conclusions clearly follow. Under the given assumptions on the ideal I, by [3, Proposition 2.6], the Rees algebra $A[It]$ satisfies (S_2). Then $A[It, t^{-1}]$ satisfies (S_2) as well (Remark 2.16), and (a) follows from the first part of Corollary 2.10. By the same Corollary 2.10, we note that $A^*(I) = \bigcup_{n\geq 1} \operatorname{Ass}(A/I^n) = \bigcup_{n\geq 1} \operatorname{Ass}(A/\overline{I^n}) = \overline{A}^*(I)$. Then, for $\mathfrak{p} \in A^*(I) = \overline{A}^*(I)$, we have $\operatorname{ht} \mathfrak{p} = \ell(I_\mathfrak{p})$ ([15, Proposition 4.1]). Since I is generated by $h+1$ elements, we also have $\ell(I_\mathfrak{p}) \leq \mu(I_\mathfrak{p}) \leq h+1$, and part (c) follows. □

Using a result of Zarzuela [20], we obtain a similar conclusion for certain ideals of analytic deviation one.

Proposition 4.4. *Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring with infinite residue field and I an ideal of height $h \geq 1$ and analytic spread $\ell(I) = h+1$. Assume that I is generically a complete intersection, the reduction number $r(I)$ is at most one, and A/I satisfies (S_1). Then*

(a) $\operatorname{Ass}(A/I^n) \subseteq \operatorname{Ass}(A/I^{n+1})$ *for all* $n \geq 1$*;*
(b) $\operatorname{ht} \mathfrak{p} \leq h+1$ *for all* $\mathfrak{p} \in A^*(I)$.

Proof. By [20, Theorem 4.4], the associated graded ring $G_I(A)$ satisfies (S_1), or equivalently, $A[It, t^{-1}]$ satisfies (S_2). Then part (a) follows from the first part of Corollary 2.10. For part (b), as in the proof of Proposition 4.3 we note that $A^*(I) = \overline{A}^*(I)$ and for $\mathfrak{p} \in \overline{A}^*(I)$ we have $\operatorname{ht} \mathfrak{p} = \ell(I_\mathfrak{p}) \leq \ell(I) = h+1$. □

In the case of a prime ideal in a regular local ring, we record the following Corollary.

Corollary 4.5. *Let $(A, \mathfrak{m})$ be a local Cohen-Macaulay ring and $\mathfrak{p}$ a prime ideal of height h such that $A_\mathfrak{p}$ is a regular local ring. Assume that either*

(i) $\mu(\mathfrak{p}) = h + 1$ *or*

(ii) $A/\mathfrak{m}$ *is infinite,* $\ell(\mathfrak{p}) = h + 1$ *and* $r(\mathfrak{p}) \leq 1$*.*

Then:

(a) $\operatorname{Ass}(A/\mathfrak{p}^n) \subseteq \operatorname{Ass}(A/\mathfrak{p}^{n+1})$ *for all* $n \geq 1$*;*

(b) $\operatorname{ht} \mathfrak{q} \leq h + 1$ *for all* $\mathfrak{q} \in A^*(\mathfrak{p})$*.*

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Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58108-6050, USA

E-mail address: `catalin.ciuperca@ndsu.edu`