MATH 421-621
FALL 2013
HOMEWORK 3

Due Wednesday, January 30, 2013.

1. (5 pt) Let A be an $R-$module and $\phi : A \longrightarrow A$ be an $R-$module homomorphism satisfying $\phi^2(a) = \phi(a)$ for all $a \in A$. Show that

$$A \cong \ker(\phi) \oplus \mathrm{im}(\phi).$$

2. Let M be an $R-$module with submodules A and B.
   a) (5 pt) If $A \cong B$ show that $M/A \cong M/B$ or give a counterexample.
   b) (5 pt) If there are $R-$module monomorphisms $\phi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ show that $A \cong B$ or give a counterexample.
   c) (5 pt) Under what conditions is $A \oplus B$ a submodule of M and when is $A \oplus B = M$?

3. Let M be an $R-$module and $I \subset R$ a proper ideal.
   a) (5 pt) Show that $IM = \{\sum_{i=1}^{n} \alpha_i m_i | \alpha_i \in I, \ m_i \in M\}$ is an $R-$submodule of M.
   b) (5 pt) Show that M/IM is an $R/I-$module.

4. (5 pt) Show that a submodule of a free $R-$module need not be free. Give an example where R is an integral domain and one where R is a finite ring.