MATH 421-621
SPRING 2013
HOMEWORK 8

Due Wednesday April 10, 2013.

1. Let R be a commutative ring with identity. We say that the characteristic of R ($\text{char}(R)$) is the smallest positive integer n such that $nr = 0$ for all $r \in R$. If no such positive n exists, we say that $\text{char}(R) = 0$.

a) (5 pt) Show that if $\mathbb{F}$ is a field, then $\text{char}(\mathbb{F})$ is either 0 or a positive prime.

b) (5 pt) Show that if $\mathbb{F}$ is a field, then $\mathbb{F}$ contains $\mathbb{Q}$ or $\mathbb{Z}_p$ for some positive prime p.

c) (5 pt) Show that if $\mathbb{F}$ is a field with finitely many elements, then $\text{char}(\mathbb{F}) = p$ and $|\mathbb{F}| = p^m$ for some positive integer m.

2. Let R be an integral domain, $\mathbb{F}$ be a field, and $f(x)$ an irreducible polynomial in $\mathbb{F}[x]$.

a) (5 pt) Show that if R is a PID, then every nonzero prime ideal of R is maximal.

b) (5 pt) Show that $\mathbb{F}[x]$ is a PID.

c) (5 pt) Show that $\mathbb{K} := \mathbb{F}[x]/(f(x))$ is a field containing (an isomorphic copy of) $\mathbb{F}$.

d) (5 pt) Show that any element of $\mathbb{K}$ is a root of a polynomial with coefficients in $\mathbb{F}$.