MATH 772
SUMMER 2006
HOMEWORK 2

Due Monday July 3, 2006.

1. (5 pt) Show that the equation $x^4+y^4 = z^2$ has no nontrivial solutions for $x, y, z \in \mathbb{N}$. Hint: show that given a solution with $x, y, z \in \mathbb{N}$, show that a "smaller" positive solution can be obtained. This technique is referred to as "infinite descent". (Note that this problem takes care of the Fermat problem for $n = 4$.)

2. (5 pt) Let R be an integral domain with quotient field K. We define the *integral closure* of R to be

$$\overline{R} = \{\alpha \in K | p(\alpha) = 0 \text{ for some monic } p(x) \in R[x]\}.$$

We say that R is integrally closed if $R = \overline{R}$ (that is, R already contains all of its integral elements from K). Prove that any UFD is integrally closed.

3. (5 pt) Let R be an integral domain with quotient field K and $p \in R$ a nonzero prime element. Show that p is also a prime element of $R[x]$.

4. (5 pt) Let F be a field extension of degree n over $\mathbb{Q}$. Suppose that $\omega \in F$ is a root of a monic polynomial in $\mathbb{Z}[x]$. Show that ω is a root of a monic polynomial in $\mathbb{Z}[x]$ of degree no more than n and, in particular, show that the minimal polynomial of ω (over $\mathbb{Q}$) may be taken to be monic and in $\mathbb{Z}[x]$.

5. (5 pt) Let d be a square-free integer. Show that the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{d})$ is given by

$$R = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \bmod(4), \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \bmod(4). \end{cases}$$