MATH 772
FALL 2011
HOMEWORK 3

Due Monday, October 24, 2011.

1. Consider the rings of integers $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{-2}]$.

a) (5 pt) Pick one of these domains and show that it is Euclidean (they both are and you may use this knowledge in the next two parts).
b) (5 pt) Find all positive primes that can be represented in the form $x^2 + 2y^2$.
c) (5 pt) Find all positive primes that can be represented in the form $x^2 - 2y^2$.

2. Consider the ring of algebraic integers R with quotient field K. One can define the norm (N) from K to $\mathbb{Q}$ via

$$N(x) = \prod_{\sigma} \sigma(x)$$

where the product ranges over all the distinct embeddings of K into $\mathbb{C}$ (and the norm on R is just the restriction of this map to R).

Perform the following tasks.

a) (5 pt) Show $N(xy) = N(x)N(y)$ (this should be easy!).
b) (5 pt) Show $N(x) = 0$ if and only if $x = 0$.
c) (5 pt) Show that the image of the norm is contained in $\mathbb{Q}$ and if $x \in R$ then $N(x) \in \mathbb{Z}$.
d) (5 pt) Show that if $x \in R$ then x is a unit of R if and only if $N(x) = \pm 1$.
e) (5 pt) For the case $R = \mathbb{Z}[\sqrt[3]{2}]$ explicitly find the norm function.
f) (5 pt) Use the previous part to show that $\mathbb{Z}[2\sqrt[3]{2}]$ is not an HFD.

3. (5 pt) Determine a quadratic ring of integers such that the inert primes are precisely the primes p such that $p \equiv 5, 11 \text{ mod}(12)$ or convince me that there is no such animal.