MATH 772
SUMMER 2006
HOMEWORK 3

Due Friday, Bastille Day, 2006.

1. Consider the rings of integers $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{-2}]$.
   a) (5 pt) Show that both of these domains are Euclidean.
   b) (5 pt) Find all positive primes that can be represented in the form $x^2 + 2y^2$.
   c) (5 pt) Find all positive primes that can be represented in the form $x^2 - 2y^2$.

2. (5 pt) (*Benko Reciprocity*) Show that if p and q are odd positive primes then

$$\binom{-p}{q} = \binom{q}{p}(-1)^{(\frac{-p-1}{2})(\frac{q-1}{2})}.$$

3. (5 pt) Consider the ring of algebraic integers R with quotient field K. One can define the norm (N) from K to $\mathbb{Q}$ via

$$N(x) = \prod_{\sigma} \sigma(x)$$

where the product ranges over all the distinct embeddings of K into $\mathbb{C}$ (and the norm on R is just the restriction of this map to R).

Perform the following tasks.
   a) (5 pt) Show $N(xy) = N(x)N(y)$ (this should be easy!).
   b) (5 pt) Show $N(x) = 0$ if and only if $x = 0$.
   c) (5 pt) Show that the image of the norm is contained in $\mathbb{Q}$ and if $x \in R$ then $N(x) \in \mathbb{Z}$.
   d) (5 pt) Show that if $x \in R$ then x is a unit of R if and only if $N(x) = \pm 1$.
   e) (5 pt) For the case $R = \mathbb{Z}[\sqrt[3]{2}]$ explicitly find the norm function.
   f) (5 pt) Use the previous part to show that $\mathbb{Z}[2\sqrt[3]{2}]$ is not an HFD.

4. (5 pt) Determine a quadratic ring of integers such that the inert primes are precisely the primes p such that $p \equiv 5, 7 \bmod(12)$ or convince me that there is no such animal.