NIM ON THE COMPLETE GRAPH

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Abstract

The game of Nim as played on graphs was introduced in [3] and extended in [4] by Masahiko Fukuyama. His papers detail the calculation of Grundy numbers for graphs under specific circumstances. We extend these results and introduce the strategy for even cycles. This paper examines a more general class of graphs by restricting the edge weight to one. We provide structural conditions for which there exist a winning strategy. This yields the solution for the complete graph.

1. Background

The general nontrivial game of Nim is a two-person combinatorial game [1] consisting of at least three piles of stones where players alternate turns, selecting first a pile from which stones will be removed, and then a strictly positive number of stones to remove. The game terminates when there are no more stones, and the winner is the player who takes the last stone or stones. Players must always remove at least one stone, and can only remove stones from a single pile during their turn.

The solution to this general game of Nim is well known and can be found in [1]. More interestingly, the solution for Nim can be applied to other two-player combinatorial games. Masahiko Fukuyama extended Nim to finite graphs by first fixing an undirected graph, assigning to each edge a positive integer, and placing a game position indicator piece at some vertex. From this indicator piece, the game begins and proceeds with alternate moves from two players according to the following rules. First, a player chooses an edge incident with the piece. The player decreases the value of this edge to any non-negative integer and moves the indicator piece to the adjacent vertex along this edge. Game play ends when a player is unable to move since the value of each edge incident with the piece equals zero. The player unable to move is the loser of the game [3]. Nim on graphs differs from the general game in that it might not be the case that all weight is removed

from the graph. We assume that a player with a winning strategy would choose to use it.

Very few results are known for Nim on graphs, adding to its appeal. Also, the strategies employed for ordinary Nim are not applicable to Nim on graphs. We will note some fundamental results of Nim on graphs following the definitions. In Section 2, we improve upon Fukuyama's result by showing the unique winning strategy for even cycles. Next in Section 3 we define a structure and strategy that leads to a first player victory. Then in Section 4 we show that the presence of this structure yields a first player win. This leads to the solution of the complete graph when each edge is given a weight of one.

Definitions. The graphs we will consider are finite and undirected with no multiple edges or loops. We will often want to label the vertices and edges. When we do, the edge between vertex v_i and v_j will be denoted e_{ij}. Graph theory terminology, including path, vertex degree, and graph isomorphism, will be assumed as found in [2].

Definition 1.1. *Given a graph G with edge set $E(G)$ and vertex set $V(G)$ we will call the non-negative integer value assigned to each $e \in E(G)$ the **weight** of the edge and denote the weight of edge e_{ij} by $\omega(e_{ij})$.*

For any graph G we assume $\omega(e_{ij}) \neq 0$ for all $e_{ij} \in E(G)$ at the start of a game. When an edge is decreased to $\omega(e) = 0$ we will delete it from the graph entirely, since it is no longer a playable edge. Given a game graph G with weight assignment $\omega_G(e)$, denote by P_1 the first player to move from the starting vertex, and denote by P_2 the player to move after P_1. The indicator piece Δ denotes the vertex from which a player is to move. We will always enumerate vertices in such a way that Δ is on v_1 at the start of a game.

Definition 1.2. *For either player and from a given position Δ on vertex v_j, we define the set of vertices to which a player may legally move to from Δ the **options** of the player. The options of player i at vertex v_j will be denoted by $O(P_i, v_j)$.*

Certainly for a vertex to exist in the set of options the incident edge must be adjacent to Δ. Thus $O(P_i, v_j) = \{v_k \in V(G) : \Delta = v_j;\ e_{jk} \in E(G);\ \omega(e_{jk}) \neq 0\}$. We will omit v_j when the position of Δ is apparent.

Definition 1.3. *We will say that a pair of P_i's options are **isomorphic** if given two options, $v_j, v_k \in O(P_i, v_i)$, there exists a graph isomorphism between v_j and its neighbors and v_k and its neighbors. We will say that two options are **identical** if in addition to being isomorphic, the options also have the same weight assignment.*

Notice that the definition of isomorphic requires that the vertices of isomorphic options have the same degree, and that there is a bijection

between the options of the vertices in the set of isomorphic options (see Example 1.4).

Example 1.4. *The options at Δ are isomorphic but not identical.*
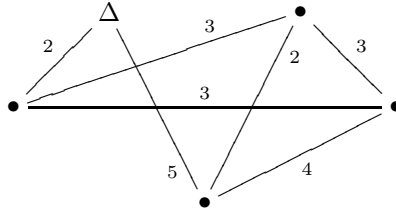
When we refer to a player winning a graph we precisely mean that a player can win a game played on that graph under the specified weight assignment and starting at a specified vertex. If no starting vertex is specified, it is true that a player can win that graph starting at any vertex. When we say that a player is on an odd path or that a player has an odd path option, we mean that there is an odd path from Δ to a vertex of degree one. However, when we say even path, we precisely mean that all options from Δ to a vertex of degree one are even paths. Notice that if G is an odd path itself, there is an odd path option at any vertex, hence there is no loss of generality by not specifying the position of Δ. Such is not the case in G is an even path, since it is possible to position Δ on vertices of G in which both options are indeed odd paths.
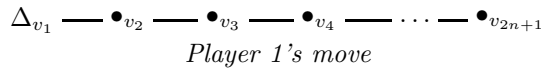
Definition 1.5. *From a particular position, if the first player to move can win for any of the second player's moves, we call this position a* **p-position**. *If the second player to move from this position can win for any of the first player's moves, we call this a* **0-position** [3].

The terms p-position and 0-position come from the *positive* and *zero* Grundy number of that particular position [1]. Grundy numbers are used heavily in many areas of two-person combinatorial game theory. Two properties especially important to keep in mind are that when a player is on a 0-position all moves are to p-positions, and that when a player is on a p-position there is always at least one move to a 0-position. Essentially, this means that a player with an advantage at the beginning can keep it with a winning strategy. Any position on an odd path is a p-position for P_1, as is any position on an odd cycle. Starting at either vertex of degree one on an even path is a 0-position for P_1, and starting at any vertex that leaves two even paths from Δ is also a 0-position for P_1. However, as noted above, if Δ started on a vertex that leaves two odd paths on this even path, the position is a 0-position for the first player.
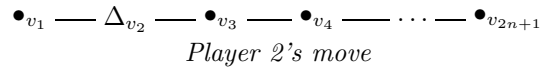
The Grundy number of a position in ordinary Nim not only told which player has an advantage at any given position, it also told that player what move to make when the Grundy number was positive. This is not the case with Nim on graphs. Knowing that you can win with Grundy number calculations does not tell you what strategies should be employed to defeat your opponent. The calculations of the Grundy numbers for trees, paths, cycles, and certain bipartite graphs can be found in [3, 4].

Paths and Cycles. The first player will win an odd path from any starting position since any vertex on an odd path has an odd path option. The strategy for the first player is to remove all weight from the edge on the odd path option. This leaves P_2 on an even path at a vertex of degree one, which as mentioned previously, is a 0-position for P_2. On the other hand, the second player will win when starting from a vertex in which every option is even path to a vertex of degree one (Example 1.6).

Example 1.6. *Starting out at v_1 we can see that the first player will move along the player's only choice of edges, once again, removing the entire edge.*

$$\Delta_{v_1} \ \rule{2em}{0.4pt} \ \bullet_{v_2} \ \rule{2em}{0.4pt} \ \bullet_{v_3} \ \rule{2em}{0.4pt} \ \bullet_{v_4} \ \rule{2em}{0.4pt} \ \cdots \ \rule{2em}{0.4pt} \ \bullet_{v_{2n+1}}$$

Player 1's move

If the first player did not remove the entire edge, it would be a faster victory for the second player.

$$\bullet_{v_1} \ \rule{2em}{0.4pt} \ \Delta_{v_2} \ \rule{2em}{0.4pt} \ \bullet_{v_3} \ \rule{2em}{0.4pt} \ \bullet_{v_4} \ \rule{2em}{0.4pt} \ \cdots \ \rule{2em}{0.4pt} \ \bullet_{v_{2n+1}}$$

Player 2's move

The second player is now on an odd path either way the second player decides to move.

Moving from paths to cycles we see that the first player to start from an odd cycle can always win. When considering an odd cycle, P_1 will move along either edge, removing all of the weight. This will leave P_2 on an even path. Since the second player to move from an even path can always win, the first player to move from an odd cycle can always win.

The strategy for either player on an odd cycle contrasts greatly with the strategy for the even cycle. In [4], Fukuyama calculated the Grundy number of even cycles. However, the calculation does not lend itself to a winning strategy for the player with the advantage. If the first player to move on an even cycle removed all weight on an edge, the first player has left the second player on an odd path. In fact, both players would want to avoid "breaking" the even cycle. Because of this, the player who is able to avoid breaking the even cycle will win. We explain the strategy for even cycles in Section 2.

Example 1.7. *In the C_4 on the left, the first player to move has the advantage. In the C_4 on the right, the second player has the advantage.*

The important part of this result is that the weights of the edges matter for even cycles. It is not difficult to show that if the weight on each edge equals one, the second player to start on an even cycle will win. This is used extensively in Section 4.

2. Strategy for Even Cycles

The Grundy number is 0 when $\omega(e) = k$ for all edges in an even cycle and for any $k \geq 0$, hence the second player to start has the advantage [4]. Consider first the strategy for P_2 when $\omega(e) = 2$ for all edges on an arbitrary even cycle.

From any starting position and for either edge, P_1 only has the choice of reducing that edge to a weight of 1 or 0. Notice that P_1 would not want to make an odd path for P_2 by reducing to 0. Thus assume without loss of generality that P_1 moves to v_2 and reduces e_{12} to $\omega(e_{12}) = 1$. Then P_2's next 0-position option is to move to v_3 leaving $\omega(e_{23}) = 1$, since moving back to v_1 requires that P_2 create an odd path for P_1. Continuing on in this way P_1 and P_2 will move to v_{2j} and $v_{2j+1}, (1 \leq j \leq n-1)$ respectively until P_1 is back at v_1 and only an even cycle with $\omega(e) = 1$ for all $e \in E(C_{2n})$, which as mentioned is a P_2 victory.

In the above case, P_1 was immediately forced to reduce the weight of an edge beyond the minimum weight of any edge. Now assume that the weights of the edges on an even cycle are arbitrary. It will still be the case that neither player wants to break the even cycle, and that the first player forced to decrease a weight below the minimum will lose. This means we can look at even cycles with arbitrary weighting assignments in the following way:

Proposition 2.1. *Assume $G = C_{2n}$ and that ω_G is some arbitrary weight assignment for G. Assume $\min_{e \in E(G)}(\omega_G(e)) = m$. Let G' be the graph formed from G under $\omega_{G'}(e) = \omega_G(e) - m$ with the same starting vertex. Then the p-positions of G are the p-positions of G' with the winning strategy for P_1 or P_2 on G following from that on G'.*

Proof. Note that G' is no longer an even cycle since at least one edge and perhaps all edges of G are deleted under $\omega_{G'}(e)$. By Proposition 6.2 in [4] which gives a calculation of the Grundy number of even cycles, the Grundy number of G is determined in part by the Grundy number of G'. As the

Grundy number of an odd path is positive and an even path is zero, the first player wins G if there is at least one odd path starting from Δ in G', and the second player wins G if all paths starting from Δ in G' are even.

To see that the strategy for playing G follows from that for G', first consider a graph with a positive Grundy number. On an odd path, we know that P_1 removes all weight on the incident edge. Since the Grundy number of G is positive, so is the Grundy number of G'. Hence G' contains an odd path. The previous paragraph implies that P_1 will move in the direction of the odd path in G' decreasing the weight of e_{12} to zero. In G, this corresponds to a move from v_1 to v_2 and a decrease of $\omega_G(e_{12})$ by $\omega_{G'}(e_{12})$ to m since $\omega_{G'}(e) = \omega_G(e) - m$ for all $e \in E(G)$.

First assume that P_2 moves back to v_1. Following this move, $\omega_G(e_{12}) = m' < m$ and we can now compare the strategy for P_1 to the strategy for some graph G'' formed from G with $\omega_{G''}(e) = \omega_G(e) - m'$ where G has been played two moves (see Example 2.2). Since G'' is an odd path of length $2n - 1$ the first player has a winning strategy.

Example 2.2. *Below are graphs of G and G' at the start of a game on even cycles. In this game $m = 2$, and since an odd path exists in G' we have a winning strategy for P_1.*
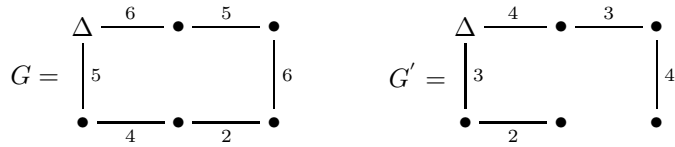
In the case that P_2 goes back to v_1 lowering the weight of the edge beyond two, we have the following graphs G and G''. Notice that in G'' there is an odd path of length $2n - 1$ since $\min_{e \in E(G)}(\omega(e)) = 1$ after the first two moves.
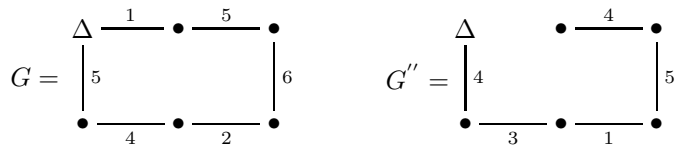
Now assume that P_2 moves to v_3 and sets $\omega(e_{23})$ to k. If $k > m$ then we know from G' that $P1$ moves back to v_2 setting $\omega(e_{23}) = m$. Since P_2 is on an even path in G', the first player will win. If $k = m$, then P_1 still has an odd path in G' and thus will win G. Finally if $k < m$ then k is the new minimum weight and there exists G'' with $\omega_{G''}(e) = \omega_G - k$ that is an odd path of length $2n - 1$ for P_1. In any case, the strategy for P_1 follows that for a graph with the lowest weight removed from every edge.

When the Grundy number of G' is zero, P_2 mimics the strategy of P_1 above.

To establish the uniqueness of this strategy, we must show that any move except one to reduce the edge weight to m on the odd path option results in a loss for the player who began on a p-position. In fact, the strategy holds at every stage of game play.

Assume P_1 begins the game on a p-position and let m be the minimum weight of any edge of $G = C_{2n}$ and $\Delta = v_1$ as before. There exists an odd path option in G', the graph formed from G under $\omega_{G'}(e) = \omega_G(e) - m$ for all $e \in E(G)$. Since taking an even path option results in a loss for P_1 by the above arguments, we assume that P_1 takes an odd path option.

Suppose that P_1 does not reduce the weight of e_{12} to m. We consider first the case when $\omega(e_{12}) = m'$ for $0 \le m' < m$ following P_1's move. With $\Delta = v_2$ and P_2's turn, we can look at a graph G'' formed from G under $\omega_{G''}(e) = \omega_G(e) - m'$. Since $m' < m$ we have that G'' is a path of length $2n - 1$. Now P_2 may move along G in the direction of the odd path in G'' reducing the edges to m' as play progresses for the win. Thus P_1 reducing any edge below m results in a loss of advantage and a P_2 win.

Now suppose that P_1 reduces e_{12} to m'' for $m < m''$ if possible, and that only one odd path option exists. If $\omega(e_{12}) = m + 1$ then we have nothing to show at this step, and if there are two odd path options, we will simply repeat this following argument a second time. With $\Delta = v_2$ and P_2's turn we will let P_2 move back to v_1 reducing the weight of e_{12} from m'' to m. In doing this, P_2 has left P_1 on an even or trivial path in the graph G' formed under the weight assignment $\omega_{G'}(e) = \omega_G(e) - m$ after two moves on G. Since this is a 0-position for P_1, we have that P_2 now holds the winning strategy. Thus using any other strategy on even cycles shifts the advantage to the player who originally started in a 0-position.

$\square$

Once the minimum weight is removed from each of the edges, it becomes clear that P_1 will win if there is an odd path option from the starting vertex. In the same way we know that P_2 will win if all first player options from the starting vertex are even paths in G' (Example 1.7).

3. A Structure Theorem

Theorem 3.1. *Let $G = K_{2,j}$ for $j \ge 1$ and $\omega(e) = 1$ for each $e \in K_{2,j}$. Assume that Δ is on a vertex in the partite set of size 2. Then P_2 will always win the $K_{2,j}$.*

Proof. We proceed by induction on j. Enumerate the vertices in the following way: Let $\Delta = v_1$ and v_2 be the other vertex in the partite set of size 2. Enumerate the vertices in the partition of size j by $v_3, v_4, \ldots, v_{j+2}$.

For $j = 1$ we have an even path. By previous work, this is a win for P_2. Similarly, for $j = 2$ we have an even cycle in which each edge has $\omega(e) = 1$ which we have also seen to be a win for P_2. Now assume that this is true for all complete $K_{2,i}$ for $i \leq j$. Consider the $K_{2,j+1}$ with Δ on v_1 in the partition of size 2. Notice that all of P_1's moves are identical since $O(P_1, \Delta = v_1) = \{v_3, v_4, \ldots, v_{j+3}\}$, all incident edges have weight 1, and $d(v_i) = 2$ for $3 \leq i \leq j + 3$.

Without loss of generality, assume that P_1 moves to v_3. Since e_{13} is now gone, as $\omega(e_{13}) = 1$ at the start, P_2 only has one move, namely to v_2. Now with Δ on v_2 and both players unable to move to v_3, we have P_1 on a $K_{2,j}$ (Figure 3.2).
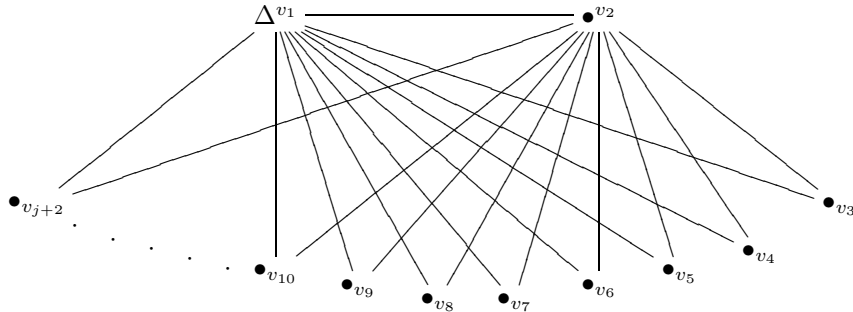
Figure 3.2.
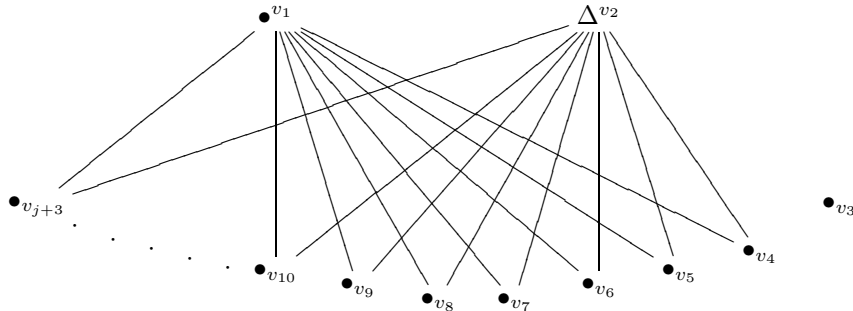
Here we have $G = K_{2,j+1}$ after the first two moves which isolates a vertex leaving a $K_{2,j}$.

By our inductive assumption, the second player will win the $K_{2,j}$. Hence P_2 wins the $K_{2,j}$ for all $j \geq 1$ and Δ on a vertex in the partition of size 2. $\qquad\square$

Now consider the $K_{2,j} + e_{12}$ with Δ still on a vertex in the partite set of size 2 and the same vertex enumeration as above.

Figure 3.3.

We will call this the SSB_j graph of order j (Figure 3.3). When the order of the graph is understood or insignificant, we will simply write SSB.

Removing e_{12} on the first move yields a $K_{2,j}$ with Δ on v_2. This lends itself to the following corollary:

Corollary 3.4. *The first player will win the SSB_j for any j when $\omega(e) = 1$ for all $e \in E(SSB_j)$ and Δ is on v_1 or v_2.*

Proof. The first player removes e_{12} and lets P_2 start on the $K_{2,j}$ with Δ on a vertex in the partite set of size two, guaranteeing P_1 the win by the previous theorem. □

It is not the case that P_1 will always win the SSB if $\omega(e) \neq 1$ for every edge. The winner can be determined by similar arguments as those for even cycles.

4. The Complete Graph

In Corollary 2.4, the first player has no option but to move back to either v_1 or v_2 since all other vertices only have degree 2. Suppose now that P_1 had more options so that the move is not forced back to v_1 or v_2 in the SSB. We continue to assume $\omega(e) = 1$ but give P_1 more options by adding edges between the vertices in the partition of size j in the SSB. We show next that additional edges do not affect a player's strategy to play the SSB when such a structure exists as a subgraph.

Lemma 4.1. *Assume that $G = K_n$ and that $\omega(e) = 1$ for all $e \in E(G)$. Then P_1 can force P_2 to move within the confines of an SSB_{n-2} contained in K_n.*

Proof. Assume $G = K_n$ with $\Delta = v_1$ and $\omega(e) = 1$ for all $e \in G$. Then all of P_1's moves are identical. Without loss of generality, assume that P_1 moves from v_1 to v_2.

Then we have $O(P_2, v_2) = \{v_3, v_4, \ldots, v_n\}$ and each option is identical. So assume without loss of generality that P_2 moves to v_3. With P_1 on $\Delta = v_3$ there are two non-isomorphic moves for P_1. One of these is to move to v_1 and the other is to move to one of the $v_4, v_5, \ldots, v_n$. Since we want to show that P_1 can move along the SSB, he would naturally choose the v_1 option.

Now $O(P_2, v_1) = \{v_4, v_5, \ldots, v_n\}$ and all of these moves are identical. Assume that P_2 moves to v_4. Then since $v_2 \in O(P_1, v_4)$ we know P_1, in keeping with the strategy to move along the SSB, will choose to move to v_2.

Continuing on in this manner we will have that $v_1 \notin O(P_2, v_2)$, $v_2 \notin O(P_2, v_1)$ since e_{12} was the first edge removed. In general, every option at every move is identical for P_2. Since $v_1 \in O(P_1, \Delta = v_i)$ for all $v_i \in O(P_2, v_2)$ and $v_2 \in O(P_1, v_j)$ for all $v_j \in O(P_2, v_1)$, P_1 is able to choose to move along the SSB.

Keeping up game play in this fashion, i.e., P_1 choosing to move to whichever of the v_1 or v_2 options exist in $O(P_1, \Delta)$ and P_2's moves identical, we will exhaust the edges incident with v_1 and v_2 leaving P_2 on an isolated vertex. Precisely, if n is even, P_2 will be stuck on v_2, and if n is odd, P_2 will be stuck on v_1. □

Notice that since P_1 never opted to use any edges outside of the SSB, the existence of those edges did not affect the strategy of P_1. We will call the technique of P_1 continually choosing to move to v_1 or v_2 from Δ the **SSB strategy** and employ this strategy in Theorem 4.4 below.

Definition 4.2. *We say two distinct vertices are **mutually adjacent** if they have the same set of neighbors and are neighbors themselves.*

Definition 4.3. *If two adjacent vertices of degree $k + 1$ have k common neighbors, we will call them **k-mutually adjacent**.*

Thus saying a graph contains two k-mutually adjacent vertices implies that the graph contains an SSB subgraph of order k. We will also speak of vertices that are k-mutually adjacent without being adjacent to each other. Notice that this implies the graph contains a $K_{2,k}$ subgraph.

Theorem 4.4. *Let G be a graph with $\omega(e) = 1$ for all $e \in E(G)$. If there exists at least two mutually adjacent vertices in G with Δ at one such vertex, then P_1 will win G.*

Proof. Assume that G is a graph of order n with $\omega(e) = 1$ for all $e \in E(G)$. Assume further that v_1 and v_2 are mutually adjacent. We proceed by induction on the k-mutual adjacency.

If v_1 and v_2 are 1-mutually adjacent and $\Delta = v_1$ then $d(v_1) = d(v_2) = 2$ and both are adjacent to some other vertex, say v_3. When P_1 moves to v_2 we have $O(P_2, v_2) = \{v_3\}$ forcing P_2's move. Then P_1 moves to v_1 for the win. Notice that this is consistent with the SSB strategy.

Assume that for all $k \leq j$ the first player to move on a graph G with at least two k-mutually adjacent vertices v_1 and v_2 and $\Delta \in \{v_1, v_2\}$ wins G by moving from v_1 to v_2 on the first move and continually choosing the v_1 or v_2 option. This implies that the second player to move from $G - e_{12}$ and $\Delta \in \{v_1, v_2\}$ wins by employing the same strategy which we are calling the SSB strategy.

Assume G is a graph of order n with $(j + 1)$-mutually adjacent vertices v_1 and v_2 for $1 < j < n - 2$ and $\Delta = v_1$. Enumerate the vertices of G in such a way that $O(P_1, v_1) = \{v_2, \ldots, v_{j+3}\}$. Suppose that P_1 moves to v_2. Then $O(P_2, v_2) = \{v_3, \ldots, v_{j+3}\}$. Without loss of generality, assume that P_2 moves to v_3. Since $v_1 \in O(P_1, v_3)$, let P_1 move to v_1. Now $O(P_2, v_1) = \{v_4, \ldots, v_{j+3}\}$. Thus we have P_2 on a j-mutually adjacent graph minus e_{12}. This means P_2 is on a complete bipartite subgraph of order j contained in G. By Theorem 3.1, the second player to start from a bipartite graph will

win, and by Lemma 4.1, since P_1 can force P_2 to move within the confines of this structure, P_1 will win this graph. Thus P_1 wins every graph G with at least two $(j+1)$-mutually adjacent vertices and Δ on a mutually adjacent vertex. $\square$

Corollary 4.5. *Assume that $G = K_n$ and that $\omega(e) = 1$ for all $e \in K_n$. Then P_1 can win the K_n for all $n > 1$.*

Proof. When $n = 2$ or 3 we have graphs that have been reduced to trivial wins for P_1. Any two vertices in the K_n are $(n-2)$-mutually adjacent. Thus for Δ at any vertex, P_1 will win the complete graph. $\square$

We have now successfully solved the problem of complete graphs when each edge has weight one. As shown, the existence of the SSB structure and appropriate starting position solves a large class of graphs. A quick check will show that the SSB strategy will not work for the complete graph and arbitrary weight assignments. However, we can show that for $n \leq 7$ the first player can win the complete graph with any weight assignment. To do this, we modify the SSB strategy slightly to account for the additional options given to the second player.

5. Acknowledgments

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