The Game of Nim on Graphs

Lindsay Merchant
North Dakota State University

September 12, 2010

Decision Theory

All forms of game theory are rooted in formal decision theory.

- Definition

  A decision problem is a problem of choosing among a set of alternatives.

Decision Theory

All forms of game theory are rooted in formal decision theory.

- Definition
  A decision problem is a problem of choosing among a set of alternatives.
  - First Condition: decision maker must know consequences of decision

Decision Theory

All forms of game theory are rooted in formal decision theory.

- Definition
  A decision problem is a problem of choosing among a set of alternatives.
  - First Condition: decision maker must know consequences of decision
  - Second Condition: existence of a preference is required

Game Theory

Game Theory is concerned with situations which have the following features:

- There must be at least 2 players.
- The game begins by one or more of the players making a choice among a number of specified alternatives.
- After the choice associated with the first move is made, a certain situation results.
- The choices made by the players may or may not become known.
- If a game is described in terms of successive choices (moves) there is a termination rule.
- Every play of a game ends in a situation.

Players

- **Definition**
  A player of a game is one who must both make choices and receive payoffs.

Players

- Definition
  A player of a game is one who must both make choices and receive payoffs.
  - Examples
    - Chance - doesn't receive payoffs
    - House - doesn't make choices
    - Slot machines

Combinatorial Game

Definition

A two-player combinatorial game requires

- Two players: P_1 and P_2
- Finitely many positions and a fixed starting position
- Rules governing moves a player can make from a given position to its options
- The player unable to move loses
- Play always ends
- Players have complete information
- No chance moves

How to play

- Start with n piles of stones.

How to play

- Start with *n* piles of stones.
- P_1 begins by selecting one of the piles. P_1 then removes any positive number of stones from this pile.
- P_2 next selects a pile and removes stones from that pile.

How to play

- Start with n piles of stones.
- P_1 begins by selecting one of the piles. P_1 then removes any positive number of stones from this pile.
- P_2 next selects a pile and removes stones from that pile.
- Play ends when there are no stones left to remove. The player who is unable to remove stones loses.

Grundy Numbers

- Constructed recursively by Patrick Michael Grundy in 1939.
- Terminal positions have a Grundy number (g-number) of 0.
- The g-number of any other position is the smallest non-negative number not already used.
- Used heavily in combinatorial game theory to describe positions

Positions

- Definition

  A P-position is a winning position for the previous player, or the player who just finished making a move.

  - P-positions are given this name for their positive g-number.

Positions

- Definition
  A P-position is a winning position for the previous player, or the player who just finished making a move.
  - P-positions are given this name for their positive g-number.
- Definition
  A 0-position is a losing position for the next player, or the player about to make a move.
  - 0-positions are given this name for their g-number of 0.

Positions

- Definition
  A P-position is a winning position for the previous player, or the player who just finished making a move.
  - P-positions are given this name for their positive g-number.
- Definition
  A 0-position is a losing position for the next player, or the player about to make a move.
  - 0-positions are given this name for their g-number of 0.
  - P-positions and 0-positions have the following properties:
    - From every 0-position, all moves are to P-positions.
    - From every P-position, there is at least one move to a 0-position.

Nim Addition

To find the Nim sum of a position in a game, suppose we start with three piles, one of size 6, one of size 4, and one of size 3.

- Write the number of stones in each pile as a binary number.
-

6	1	1	0
4	1	0	0
3		1	1

Nim Addition

To find the Nim sum of a position in a game, suppose we start with three piles, one of size 6, one of size 4, and one of size 3.

- Write the number of stones in each pile as a binary number.
- Add the piles, modulo 2 in each column.
- This nonnegative number is the Nim sum of the piles.

6	1	1	0
4	1	0	0
3		1	1

6	1	1	0
4	1	0	0
3		1	1
	0	0	1

Strategy to Win

- Find the Nim sum of the piles. The positive remainder is the number of stones you must remove from one of the piles.

6	1	1	0
4	1	0	0
3		1	1
	0	0	1

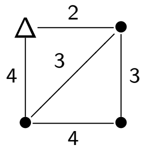
Strategy to Win

- Find the Nim sum of the piles. The positive remainder is the number of stones you must remove from one of the piles.
- This leaves your opponent with a g-number of 0, thus inflicting a 0-position.

6	1	1	0
4	1	0	0
3		1	1
	0	0	1

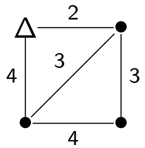
6	1	1	0
4	1	0	0
2		1	0
	0	0	0

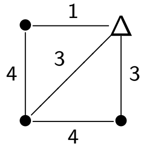
How to Play

- Start with a graph G.
  For each $e \in E(G)$ define a map $\omega(e) : E(G) \to \mathbb{N}$ that assigns a *weight* to each edge of G.
  Fix a starting position at some vertex of G represented by Δ.
- 

How to Play

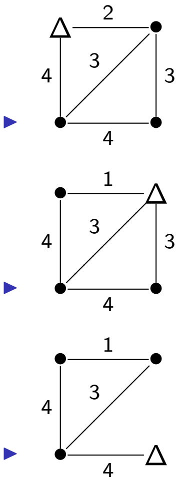
- Start with a graph G.
  For each $e \in E(G)$ define a map $\omega(e) : E(G) \to \mathbb{N}$ that assigns a *weight* to each edge of G.
  Fix a starting position at some vertex of G represented by Δ.
- P_1 moves along any edge incident with Δ to another vertex decreasing the weight of that edge to a strictly nonnegative number.





How to Play

- Start with a graph G.
  For each $e \in E(G)$ define a map $\omega(e) : E(G) \rightarrow \mathbb{N}$ that assigns a *weight* to each edge of G.
  Fix a starting position at some vertex of G represented by Δ.
- P_1 moves along any edge incident with Δ to another vertex decreasing the weight of that edge to a strictly nonnegative number.
- P_2 moves from this new Δ to a vertex adjacent.



How to Play

- Once the weight of an edge equals zero, neither player may move across it.

How to Play

- Once the weight of an edge equals zero, neither player may move across it.
- Play continues in this back and forth manner until a player gets stuck at a vertex.

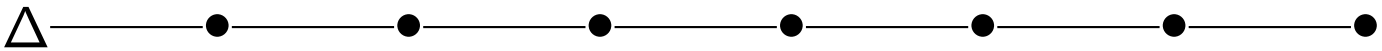
How to Play

- Once the weight of an edge equals zero, neither player may move across it.
- Play continues in this back and forth manner until a player gets stuck at a vertex.
- Not necessarily the case that all edges are removed from the graph before the game ends.

Odd Paths

When we refer to the length of the path, we consider the number of edges. Suppose we start at the end of a path first. Also assume the weights on the edges are arbitrary.

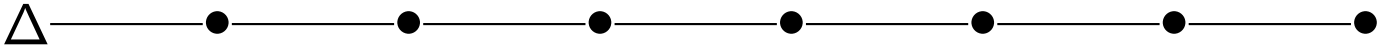
- P_1's move



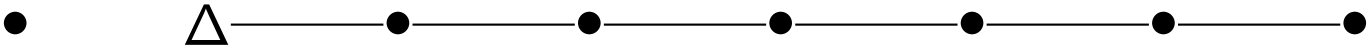
Odd Paths

When we refer to the length of the path, we consider the number of edges. Suppose we start at the end of a path first. Also assume the weights on the edges are arbitrary.

- P_1's move



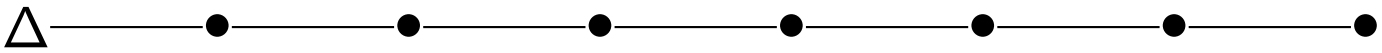
- P_2's move



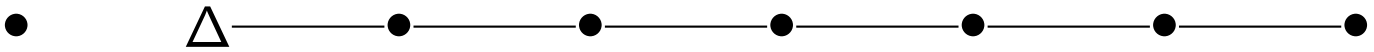
Odd Paths

When we refer to the length of the path, we consider the number of edges. Suppose we start at the end of a path first. Also assume the weights on the edges are arbitrary.

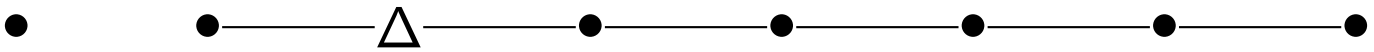
- P_1's move



- P_2's move



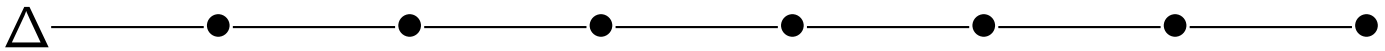
- P_1's move



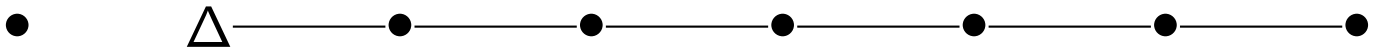
Odd Paths

When we refer to the length of the path, we consider the number of edges. Suppose we start at the end of a path first. Also assume the weights on the edges are arbitrary.

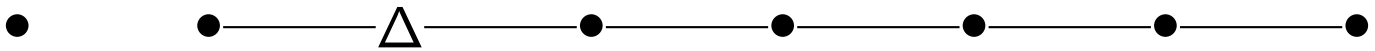
- P_1's move



- P_2's move



- P_1's move

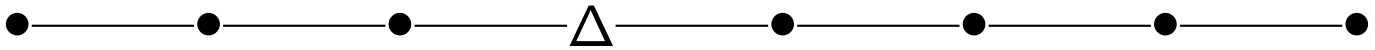


- P_2 loses

Odd Paths

Now suppose the position in the odd path is arbitrary.

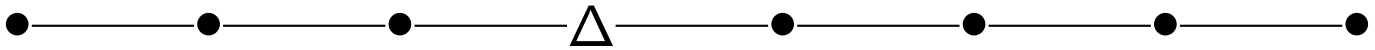
- P_1's move



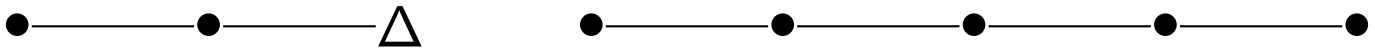
Odd Paths

Now suppose the position in the odd path is arbitrary.

- P_1's move



- P_2's move



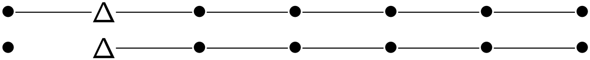
Even Paths

- P_1's move

Even Paths

- P_1's move

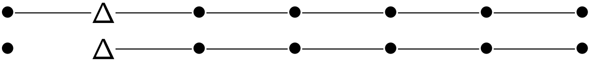
- P_2's move



Even Paths

- P_1's move

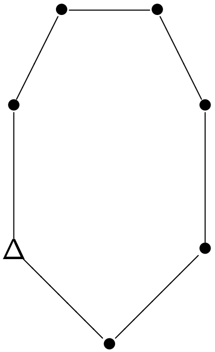
- P_2's move



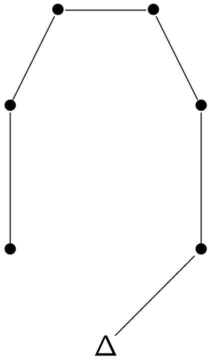
- In either situation for P_2 there is a path of odd length, thus resulting in a P_2 win with any weighting assignment.

Odd Cycles

Continue to assume that the weight assignment is arbitrary



P_1's move



P_2's move

Nim on Graphs I

Results:

- Assumes all game graphs are bipartite and P_2's vertices have degree 2.
- Finds whether a given position is a P-position or 0-position for such graphs.
- Solves the problem of finding a Grundy number for odd and even paths in the process.

Nim on Graphs II

Results:

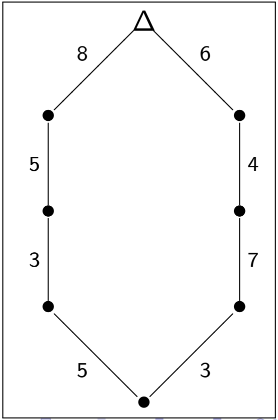
- Finds the g-number for bipartite graphs with matchings and without alternating cycles.
- Determines whether given positions are P-positions and 0-positions for such graphs.
- Finds the g-number for cycles and trees completely.

Do Grundy Numbers Matter?

- Previous results due to Fukuyama only give g-numbers in terms of relative positions.
- In contrast to regular Nim, knowing the g-number does not tell you what move to make.
- No convincing evidence that g-numbers will matter for Nim on Graphs when played in this fashion.

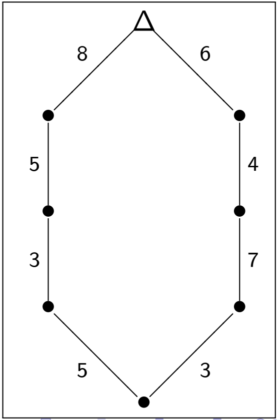
Strategy for Even Cycles

- Suppose $G = C_{2n}$ is arbitrarily weighted with starting piece Δ.



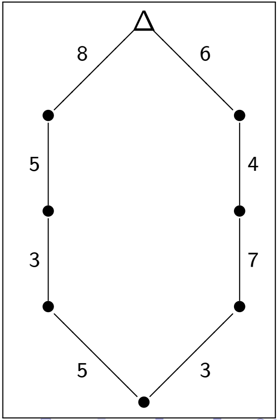
Strategy for Even Cycles

- Suppose $G = C_{2n}$ is arbitrarily weighted with starting piece Δ.
- Begin by finding $\min(\omega(e))$ amongst all $e \in E(G)$.



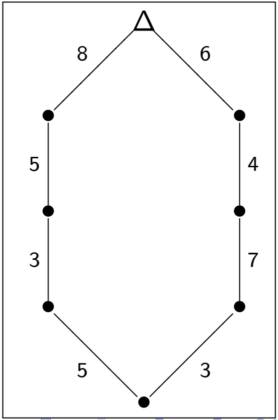
Strategy for Even Cycles

- Suppose $G = C_{2n}$ is arbitrarily weighted with starting piece Δ.
- Begin by finding $\min(\omega(e))$ amongst all $e \in E(G)$.
- The two distances from Δ to the vertices incident with $\min_{e \in E(G)}(\omega(e))$ determine the winner of the game.



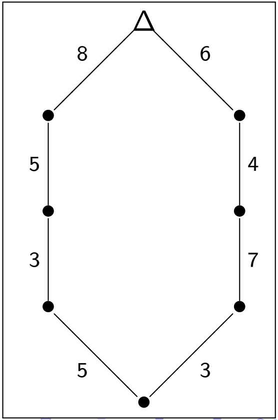
Strategy for Even Cycles

- If there is at least one odd path from Δ to a vertex incident with an edge of minimum weight, then P_1 will win.



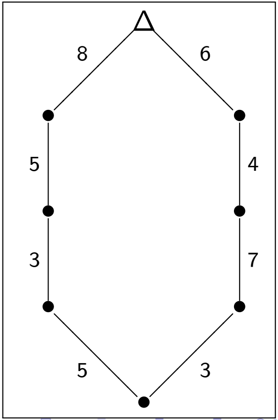
Strategy for Even Cycles

- If there is at least one odd path from Δ to a vertex incident with an edge of minimum weight, then P_1 will win.
- If all paths are even from Δ to a vertex incident with an edge of minimum weight, then P_2 will win.



Strategy for Even Cycles

- If there is at least one odd path from Δ to a vertex incident with an edge of minimum weight, then P_1 will win.
- If all paths are even from Δ to a vertex incident with an edge of minimum weight, then P_2 will win.
- In both cases, the strategy for either player is to move in the direction of the edge with lowest weight, decreasing the weight of each edge to $\min(\omega(e))$.



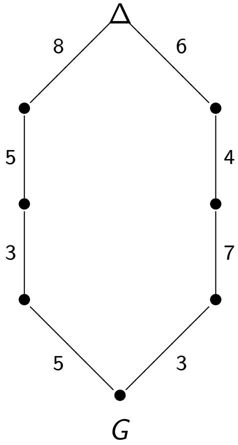
Strategy for Even Cycles

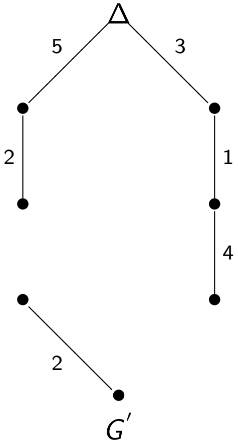
Theorem

Assume $G = C_{2n}$ and that ω_G is some arbitrary weight assignment for G. Assume $\min_{e \in E(G)}(\omega_G(e)) = m$. Let G' be the graph formed from G under $\omega_{G'}(e) = \omega_G(e) - m$ with the same starting vertex. Then the p-positions of G are the p-positions of G' with the winning strategy for P_1 or P_2 on G following from that on G'.

Strategy for Even Cycles

Proof.





Strategy for Even Cycles

Proof.

- The fact that the positional values of G and G' are the same follows from results by Fukuyama
- Assume that there is a path of odd length from Δ to a vertex incident with $\min(\omega(e))$

□

Strategy for Even Cycles

Proof.

- The fact that the positional values of G and G' are the same follows from results by Fukuyama
- Assume that there is a path of odd length from Δ to a vertex incident with $\min(\omega(e))$
- Notice the first player to break the cycle will loose
- Let P_1 employ the same strategy on G as he would on G', that is to move in the direction of the odd path, decreasing the weight of each edge of G by that of the same edge on G'

□

Strategy for Even Cycles

Proof.

(continued)

- P_2 will be the first player forced to decrease an edge of G below m

□

Strategy for Even Cycles

Proof.

(continued)

- P_2 will be the first player forced to decrease an edge of G below m
- After P_2's move that decreases an edge weight below m, we can consider a new graph G'' formed from the current state of G less that new lowest weight on each edge

□

Strategy for Even Cycles

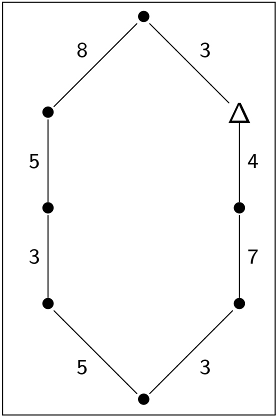
Proof.

(continued)

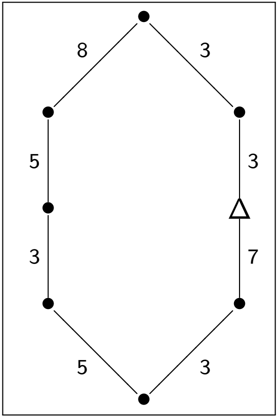
- P_2 will be the first player forced to decrease an edge of G below m
- After P_2's move that decreases an edge weight below m, we can consider a new graph G'' formed from the current state of G less that new lowest weight on each edge
- G'' is a path of length $2n - 1$, which is an odd path and a P_1 win. Play continues in this manner until P_2 is forced to remove an edge entirely, thus creating an odd path in G

□

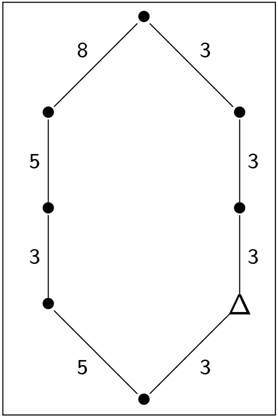
Strategy for Even Cycles



P_2's turn



P_1's turn



P_2's turn

Bipartite graphs with $\omega(e) = 1$

- Theorem

  Let $G = K_{2,j}$ for $j \geq 1$ and $\omega(e) = 1$ for each $e \in K_{2,j}$. Assume that Δ is on a vertex in the partite set of size 2. Then P_2 will always win the $K_{2,j}$.

  Proof.

Bipartite graphs with $\omega(e) = 1$

- Theorem

  Let $G = K_{2,j}$ for $j \geq 1$ and $\omega(e) = 1$ for each $e \in K_{2,j}$. Assume that Δ is on a vertex in the partite set of size 2. Then P_2 will always win the $K_{2,j}$.

  Proof.

  - Simple induction argument; base case $j = 1$ is a path of length 2 with all edges weight 1, which we've seen is a P_2 win. Notice for $j = 2$ we have an even cycle with the trivial even path to an edge of minimum weight.

Bipartite graphs with $\omega(e) = 1$

- Theorem
  Let $G = K_{2,j}$ for $j \geq 1$ and $\omega(e) = 1$ for each $e \in K_{2,j}$. Assume that Δ is on a vertex in the partite set of size 2. Then P_2 will always win the $K_{2,j}$.

  Proof.

  - Simple induction argument; base case $j = 1$ is a path of length 2 with all edges weight 1, which we've seen is a P_2 win. Notice for $j = 2$ we have an even cycle with the trivial even path to an edge of minimum weight.
  - Assume true for $j \leq n - 1$. Start with $\Delta = v_1$ and v_2 the other vertex in the partite set of size 2. Suppose there are n vertices in the other partite set.

Bipartite graphs with $\omega(e) = 1$

Proof.

- P_1's options are isomorphic to one of the vertices amongst $v_3, \ldots, v_{j+2}$.

□

Bipartite graphs with $\omega(e) = 1$

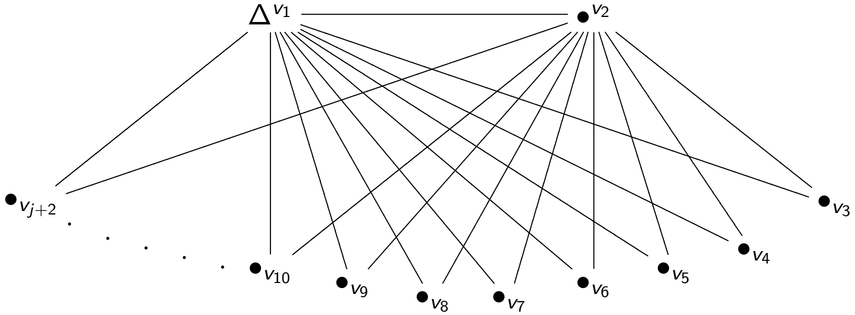
Proof.

- P_1's options are isomorphic to one of the vertices amongst $v_3, \ldots, v_{j+2}$.
- P_2 is forced to move to v_2. The resulting graph is the $K_{2,n-1}$ which by inductive assumption P_2 will win.

□

The *SSB* subgraph

- Assume $\omega(e) = 1$ for all edges.
- Construct the SSB_j graph of order $j + 2$ from the $K_{2,j}$ with an additional edge between the vertices in the partite set of size 2.



The *SSB* subgraph

- Corollary

  The first player will win the SSB_j *for any* j *when* $\omega(e) = 1$ *for all* $e \in E(SSB_j)$ *and* Δ *is on* v_1 *or* v_2*.*

The *SSB* subgraph

- Corollary

  The first player will win the SSB_j for any j when $\omega(e) = 1$ for all $e \in E(SSB_j)$ and Δ is on v_1 or v_2.

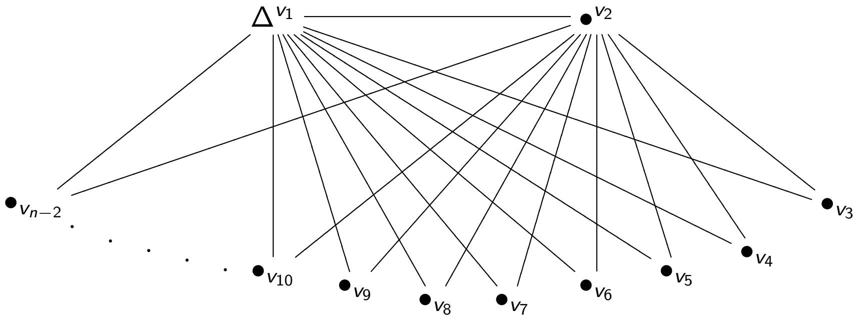
- Proof.

  The first player removes e_{12} and lets P_2 start on the $K_{2,j}$ with Δ on a vertex in the partite set of size two, guaranteeing P_1 the win by the previous theorem. □

The *SSB* subgraph

Lemma

Assume that $G = K_n$ and that $\omega(e) = 1$ for all $e \in E(G)$. Then P_1 can force P_2 to move within the confines of an SSB_{n-2} contained in K_n.



The *SSB* strategy

The Complete Graph

- Definition
  We say two distinct vertices are **mutually adjacent** if they have the same set of neighbors and are neighbors themselves.

The Complete Graph

- Definition
  We say two distinct vertices are **mutually adjacent** if they have the same set of neighbors and are neighbors themselves.

- Definition
  If two adjacent vertices of degree $k + 1$ have k common neighbors, we will call them **k-mutually adjacent**.

The Complete Graph

- Definition
  We say two distinct vertices are **mutually adjacent** if they have the same set of neighbors and are neighbors themselves.
- Definition
  If two adjacent vertices of degree $k+1$ have k common neighbors, we will call them **k-mutually adjacent**.
  - Thus saying a graph contains two k-mutually adjacent vertices implies that the graph contains an *SSB* subgraph of order k. We will also speak of vertices that are k-mutually adjacent without being adjacent to each other. Notice that this implies the graph contains a $K_{2,k}$ subgraph.

Main Theorem

Theorem

Let G be a graph with $\omega(e) = 1$ for all $e \in E(G)$. If there exists at least two mutually adjacent vertices in G with Δ at one such vertex, then P_1 will win G.

Proof.

- If there are at least two mutually adjacent vertices in G then there is an SSB subgraph in G.

Main Theorem

Theorem

Let G be a graph with $\omega(e) = 1$ for all $e \in E(G)$. If there exists at least two mutually adjacent vertices in G with Δ at one such vertex, then P_1 will win G.

Proof.

- If there are at least two mutually adjacent vertices in G then there is an *SSB* subgraph in G.
- By the previous Lemma, we know that P_1 can keep P_2 within the confines of the *SSB* since we assumed that Δ was at one of these mutually adjacent vertices.

Main Theorem

Theorem

Let G be a graph with $\omega(e) = 1$ for all $e \in E(G)$. If there exists at least two mutually adjacent vertices in G with Δ at one such vertex, then P_1 will win G.

Proof.

- If there are at least two mutually adjacent vertices in G then there is an *SSB* subgraph in G.
- By the previous Lemma, we know that P_1 can keep P_2 within the confines of the *SSB* since we assumed that Δ was at one of these mutually adjacent vertices.
- We know that P_1 wins the *SSB* of any order, hence P_1 wins G.

The Complete Graph

- Corollary

  P_1 wins the complete graph of any order when each edge has weight one.

The Complete Graph

- Corollary

  P_1 wins the complete graph of any order when each edge has weight one.

- Proof.

  When $n = 2$ or 3 we have graphs that have been reduced to trivial wins for P_1. Any two vertices in the K_n are $(n-2)$-mutually adjacent. Thus for Δ at any vertex, P_1 will win the complete graph. □

On-going Research

- We now have a result for cubic graphs when $\omega(e) = 1$:
  - P_1 wins the Q_{2n+1} for any $n \geq 0$
  - P_2 wins the Q_{2n} for any $n \geq 0$

On-going Research

- We now have a result for cubic graphs when $\omega(e) = 1$:
  - P_1 wins the Q_{2n+1} for any $n \geq 0$
  - P_2 wins the Q_{2n} for any $n \geq 0$
- Current research includes the complete graph with arbitrary weight. Results up to the K_7 show that P_1 will win.

Thanks!

References

- Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy.
  Winning ways for your mathematical plays. Vol. 1.
  A K Peters Ltd., Natick, MA, second edition, 2001.

- Masahiko Fukuyama.
  A Nim game played on graphs.
  Theoret. Comput. Sci., 304(1-3):387–399, 2003.

- Masahiko Fukuyama.
  A Nim game played on graphs. II.
  Theoret. Comput. Sci., 304(1-3):401–419, 2003.