

ON FREE DISCRETE SUBGROUPS OF $Diff(I)$

AZER AKHMEDOV

ABSTRACT: We prove that a free group \mathbb{F}_2 admits a faithful discrete representation into $Diff_+^1[0, 1]$. We also prove that \mathbb{F}_2 admits a faithful discrete representation of bi-Lipschitz class into $Homeo_+[0, 1]$. Some properties of these representations have been studied. In the last section of the paper we raise several questions.

INTRODUCTION

In recent decades and especially in recent years, some remarkable papers have appeared which are devoted to the study of finitely generated subgroups of $Diff_+^1[0, 1]$ (see [B], [C], [FF1], [FF2], [FS], [G], [N1], [N2], [N3], [Ts], [Y] for some of the most recent developments). In contrast, discrete subgroups of the group $Diff_+^1[0, 1]$ are much less studied. Very little is known in this area especially in comparison with the very rich theory of discrete subgroups of Lie groups which has started in the works of F.Klein and H.Poincaré in the 19th century, and has experienced enormous growth in the works of A.Selberg, A.Borel, G.Mostow, G.Margulis and many others in the 20th century. Many questions which are either very easy or have been studied a long time ago for (discrete) subgroups of Lie groups remain open in the context of the infinite-dimensional group $Diff_+^1[0, 1]$ and its relatives. In this paper, we address a question about the existence of discrete faithful representations of non-abelian free groups into the group $Diff_+^1[0, 1]$.

We assume the usual topology on the group $Diff_+^1[0, 1]$ given by the standard metric of $C^1[0, 1]$. We will denote this metric by d_1 .

Theorem 1. A free group \mathbb{F}_2 admits a faithful discrete representation into $Diff_+^1[0, 1]$.

We will also be interested in discrete subgroups of $Homeo_+[0, 1]$ - the group of orientation preserving homeomorphisms of the closed interval. Here, the metric comes from the sup norm of the Banach space $C[0, 1]$. For $f \in C[0, 1]$ we will denote $\|f\|_0 = \sup_{x \in [0, 1]} |f(x)|$.

Theorem 2. A free group \mathbb{F}_2 admits a faithful discrete representation into $Homeo_+[0, 1]$. Moreover,

- a) the representation can be chosen from the class $C^1(0, 1) \cap BiLip[0, 1]$.

b) for any non-empty open neighborhood Ω of the identity in $Homeo_+[0, 1]$, the generators of the faithful discrete representation of \mathbb{F}_2 can be chosen from Ω .

Here, $BiLip[0, 1]$ denotes the set of all bi-Lipschitz functions from the closed interval $[0, 1]$ into itself.

PROOFS OF MAIN THEOREMS

In this section we will prove Theorem 1 and Theorem 2.

In the free group \mathbb{F}_2 we will fix the left-invariant Cayley metric with respect to standard generating set, and denote it by $|\cdot|$. The following notions will be useful.

Definition 1. Let W be a reduced word in the alphabet of the standard generating set of the free group \mathbb{F}_2 . We say that a reduced word U is a *suffix* of W , if $W = U_1U$ where U_1 is a reduced word, and $|W| = |U_1| + |U|$. We also say that a reduced word V is a *prefix* of W , if $W = VV_1$ where V_1 is a reduced word, and $|W| = |V| + |V_1|$.

Proof of Theorem 1.

Let $I_n = (\frac{1}{2n+1}, \frac{1}{2n})$ for any $n \in \mathbb{N}$ and let $C > 0$.

We will build two maps $f, g \in Diff_+^1[0, 1]$ such that the group $\Gamma_{f,g}$ generated by them is isomorphic to \mathbb{F}_2 and satisfies the following condition:

(\star) for all $g_1, g_2 \in \Gamma_{f,g}, g_1 \neq g_2$, the inequality $sup_{t \in [0,1]} |g_1'(t) - g_2'(t)| > C$ is satisfied.

Let $\pi_n = (U_n, V_n), n \geq 1$ be a sequence of pairs of words (elements) in \mathbb{F}_2 satisfying the following conditions:

(a1) $U_n \neq V_n$ for all $n \geq 1$.

(a2) $|U_n| \geq |V_n|$ for all $n \geq 1$.

(a3) if $m > n$ then $|U_m| \geq |U_n|$.

(a4) if $m > n, |U_m| = |U_n|$ then $|V_m| \geq |V_n|$.

(a5) $U_n \neq 1$, for all $n \geq 1$.

(a6) if $U, V \in \mathbb{F}_2, U \neq 1, |U| \geq |V|$ then there exists $n \in \mathbb{N}$ such that $U = U_n, V = V_n$.

(a7) if $m \neq n$ then $\pi_m \neq \pi_n$.

For every $n \in \mathbb{N}$, the longest common suffix of U_n and V_n will be denoted by W_n and we let $s_n = |W_n|$.

Let also $m_n = \text{Card}\{k \mid \pi_k = (U_k, V_k), |U_k| = n\}$ for all $n \geq 1, m_0 = 0$. Notice that m_n grows exponentially as $n \rightarrow \infty$.

Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a sequence of positive real numbers such that

$$(b1) \lim_{r \rightarrow \infty} \alpha_r = 0$$

(b2) for every $r \in \mathbb{N}, s \in \{0, 1, \dots, r-1\}$, the inequality

$$(1 + \alpha_r)^s [(1 + \alpha_r)^{r-s} - 1] > C$$

is satisfied.

[Notice that such a sequence α exists, e.g. $\alpha_1 = C + 1, \alpha_r = \sqrt{\frac{C+1}{r-1}}, r \geq 2$.]

Let also $\beta = (\beta_1, \beta_2, \dots)$ be a sequence such that $\beta_i = \alpha_j$ for all $m_1 + \dots + m_{j-1} < i \leq m_1 + \dots + m_{j-1} + m_j$. We notice that $\lim_{n \rightarrow \infty} \beta_n = 0$; moreover, for every $n \in \mathbb{N}, \beta_n = \alpha_{i(n)}$ where $i(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Now, for any natural n , let x_0^n be the midpoint of the interval I_n , $s = s_n$, and let f, g be defined in the interval I_n such that

$$(c1) f(x) = g(x) = x \text{ for all } x \in \{\frac{1}{2n+1}, \frac{1}{2n}\}$$

$$(c2) f'(x) \in [\frac{1}{1+\beta_n+\frac{1}{n}}, 1 + \beta_n + \frac{1}{n}], \text{ for all } x \in I_n.$$

$$(c3) f'(x) = g'(x) = 1 \text{ for all } x \in \{\frac{1}{2n+1}, \frac{1}{2n}\}$$

(c4) if $|U_n| = r, U_n = a_r a_{r-1} \dots a_s \dots a_1, a_i \in \{f, g, f^{-1}, g^{-1}\}, 1 \leq i \leq r, U_n(k) = a_k \dots a_1, 0 \leq k \leq r-1$ then $a'_{k+1}(U_n(k)(x_0^n)) = 1 + \beta_n$.

(c5) if $|V_n| = m, V_n = b_m b_{m-1} \dots b_1, b_i \in \{f, g, f^{-1}, g^{-1}\}, 1 \leq i \leq m$, and if $V_n(k) = b_k \dots b_1, m-1 \geq k \geq s$ then the equality $b'_{k+1}(V_n(k)(x_0^n)) = 1$ holds.

Now, if $x \in [0, 1] \setminus \sqcup_{n \in \mathbb{N}} I_n$, we set $f(x) = g(x) = x$ (hence $f'(x) = g'(x) = 1$).

Then the functions f, g will belong to $\text{Diff}_+^1[0, 1]$. Moreover, for any $n \geq 1$, by Chain Rule, we have

$$U'_n(x_0^n) = (1 + \beta_n)^r, V'_n(x_0^n) = (1 + \beta_n)^s 1^{m-s} = (1 + \beta_n)^s$$

Since $\beta_n = \alpha_{i(n)}$ and $i(n) = r$, the inequality $|(U_n(f, g))'(x_0^n) - (V_n(f, g))'(x_0^n)| > C$ follows from condition (b2). \square

Remark 1. We indeed prove more than discreteness; the inequality

$$\sup_{t \in [0, 1], g \in \mathbb{F}_2 \setminus \{1\}} |g'(t) - 1| \geq C > 0$$

would suffice for discreteness. By proving more general inequality

$$\sup_{t \in [0,1], g_1, g_2 \in \mathbb{F}_2, g_1 \neq g_2} |g'_1(t) - g'_2(t)| \geq C > 0$$

we show that the representation is *uniformly discrete*. Since the metric in $Diff^1_+[0, 1]$ is not left-invariant, discreteness does not necessarily imply uniform discreteness.

Remark 2. It is clear from the proof that the functions $f(t)$ and $g(t)$ can be chosen from an arbitrary non-empty open neighborhood of the identity. This is contrary to the case of connected Lie groups: the *Margulis Lemma* states that any connected Lie group G possesses a non-empty open neighborhood U of the identity such that any discrete subgroup of G generated by elements from U is nilpotent (see [R]). Thus we have shown that the Margulis Lemma does not hold for the group $Diff^1_+[0, 1]$.

It is easy to put the main idea of the proof of Theorem 1 in words: we take all pairs (U_n, V_n) in the free group \mathbb{F}_2 that are interesting to us and enumerate them with some care, [conditions (a1) – (a7)]. For simplicity, let us also assume that $V_n = 1, n \geq 1$. Then we choose countable pairwise disjoint open subintervals $I_1, I_2, \dots, I_n, \dots$ of $[0, 1]$ which are accumulating to the left endpoint of $[0, 1]$, (I_i is on the left side of I_j for all $i > j$). Then, on each of the subintervals we arrange the maps f, g such that $\sup_{x \in I_n} |f'(x) - 1|$ and $\sup_{x \in I_n} |g'(x) - 1|$ converge to zero as $n \rightarrow \infty$ while for each midpoint $x_n \in I_n$ we have $U'_n(x_n) > C$.

To satisfy this condition, one notices that the word U_n has length at least $\log(n)$ which goes to infinity as n grows. Then, since $U'_n(x_n)$ is the product of $\log(n)$ derivatives we can have this product to be bigger than C yet each of the factor stay close to 1. (and converge to 1 as n goes to infinity). For fixed n , each of these conditions imposes only finitely many conditions on f and g in I_n , and for the next pair we go to a different interval I_{n+1} , hence we have no obstruction left to the existence of discrete \mathbb{F}_2 of C^1 class.

However, because of the slow growth of $\log(n)$, and because the lengths of intervals of I_n converge to zero faster than $1/n$, it is easy to see that this construction will not work in C^2 class. In fact, as Danny Calegari pointed out, it will not work in any $C^{1+\epsilon}$ class for any $\epsilon > 0$; imposing the same condition will blow-up the Holder norm. So one cannot achieve higher regularity of representations by taking care of different pairs in disjoint areas of the closed interval $[0, 1]$. If we

want to mix fields of actions for different pairs, we need to take some cautions.

Now we will prove Theorem 2. We need the following definitions.

Definition 2. For open subintervals $I, J \subset (0, 1)$ we say $I < J$ if any element I is less than any element of J .

Definition 3. A two-sided sequence $\{I_n\}_{n \in \mathbb{Z}}$ of open subintervals of $(0, 1)$ is called a *chain* if $I_n < I_{n+1}$ for all $n \in \mathbb{Z}$.

Proof of Theorem 2. Let $\epsilon > 0$, and let A_n, B_n , $n \in \mathbb{Z}$ be open subintervals of $(0, 1)$ such that

- (i) the two sided sequence $\{A_n, B_n\}_{n \in \mathbb{Z}}$ is a chain of subintervals. [i.e. we have $\dots < A_{-1} < B_{-1} < A_0 < B_0 < A_1 < B_1 < A_2 < \dots$]
- (ii) for all $n \in \mathbb{Z}$, and for all $i \in \{1, 2, 3, 4\}$ we have $f^i(A_n) \subseteq B_n, f^{-i}(A_n) \subseteq B_{n-1}$.
- (iii) for all $n \in \mathbb{Z}$,

$$g(B_n) \subseteq A_{n+1}, g^{-1}(B_n) \subseteq A_n$$

- (iv) for all $n \in \mathbb{Z}$, the inequality $\sup_{x \in A_n, y \in A_{n+2}} |x - y| < \epsilon$ holds.

It is straightforward to choose $f, g \in \text{Homeo}_+[0, 1]$ satisfying conditions (i)-(iv).

Now, let $A = \cup_{n \in \mathbb{Z}} A_n, B = \cup_{n \in \mathbb{Z}} B_n$. Notice that by conditions (i)-(ii)

$$f^i(A) \subseteq B \quad \forall i \in \{-4, -3, -2, -1, 1, 2, 3, 4\}, \quad g^i(B) \subseteq A, \quad \forall i \in \{-1, 1\}$$

This allows us to use a ping-pong argument.

The ping-pong argument is usually used to guarantee existence of free subgroups, here we will be using it also to satisfy discreteness (which is natural). Using the ping-pong lemma, we will show that, assuming conditions (i)-(iv), if $U(f, g), V(f, g)$ are reduced words satisfying conditions

(1) $U(f, g) = f^2 U_0(f, g) f^2, V(f, g) = f V_0(f, g) f$ where $U_0(f, g), V_0(f, g)$ are both non-empty reduced words starting and ending in letter g ,

(2) none of the letters $\{f, g\}$ occur with exponent other than $\{-1, 1\}$ in $U_0(f, g)$ and in $V_0(f, g)$,

then $U(f, g), V(f, g)$ actually generate a free subgroup isomorphic to \mathbb{F}_2 in $\text{Homeo}_+[0, 1]$. We will have that this subgroup (which we will denote by Γ) is discrete.

Let $W(U, V)$ be any reduced non-trivial word in the alphabet $\{U = U(f, g), V = V(f, g), U^{-1} = U(f, g)^{-1}, V^{-1} = V(f, g)^{-1}\}$. Then in the alphabet $\{f, g, f^{-1}, g^{-1}\}$ the word W ends with either f or f^{-1} .

Let x_0 be the midpoint of A_0 .

We notice that $f^i(A) \subseteq B$ for all $i \in \{-4, -3, -2, -1, 1, 2, 3, 4\}$. Furthermore, $g^{\pm 1}(B) \subseteq A$. Then by a standard ping-pong argument, we have that

$$W(x_0) = W(U(f, g), V(f, g))(x_0) \notin A_0$$

hence $W \neq 1$ in Γ , and $\|W\|_0 \geq |A_0|/2$.

We now consider the general case of arbitrary distinct $h_1, h_2 \in \Gamma$. Let $h_1 = W_1(U, V), h_2 = W_2(U, V)$ be two distinct reduced words in the alphabet $\{U, V, U^{-1}, V^{-1}\}$. Then we can write $W_2 = WW_1$ where $W = W(U, V) = W(U(f, g), V(f, g))$.

Since $W_1 = W_1(U(f, g), V(f, g))$ is a bijective map from $[0, 1]$ onto $[0, 1]$, there exists $z \in [0, 1]$ such that $W_1(z) = x_0$. Then $W_2(z) = W(W_1(z)) = W(x_0) \notin A_0$.

Then we have $|W_1(z) - W_2(z)| = |x_0 - W(x_0)| > |A_0|/2$. Thus we established that the non-abelian free subgroup generated by U and V is discrete.

For the claim b), suppose Ω contains a ball of radius r , and $M = \max\{|U|, |V|\}$. Then by condition (iv), $\max\{\|U\|_0, \|V\|_0\} < \epsilon M$. Since ϵ is arbitrary we can choose it to be such that $M\epsilon < r$, and hence we obtain the claim b).

For the claim a), we may choose a sufficiently large natural number N , and further assume that

$$A_n = \left(\frac{1}{5(|n|+1)}, \frac{1}{5|n|+4} \right), B_n = \left(\frac{1}{5|n|+4}, \frac{1}{5|n|} \right), \forall n \leq -N$$

and

$$A_n = \left(1 - \frac{1}{5n}, 1 - \frac{1}{5n+1} \right), B_n = \left(1 - \frac{1}{5n+1}, 1 - \frac{1}{5(n+1)} \right), \forall n \geq N$$

(and we choose $A_{-N+1}, B_{-N+1}, \dots, A_{N-1}, B_{N-1}$ to be arbitrary open non-empty intervals such that conditions (i) and (iv) hold). Then it is straightforward to choose $f, g \in \text{Homeo}_+[0, 1]$ such that $f \in C^1[0, 1], g \in C^1(0, 1)$, and g is a bi-Lipschitz function with Lipschitz constant at most 5 in $[0, \frac{1}{5N}]$ and in $[1 - \frac{1}{5N}, 1]$. Then for any word W in the free group \mathbb{F}_2 , the function $W(U(f, g), V(f, g))$ will be a bi-Lipschitz function of class $C^1(0, 1)$. \square

Remark 3. We would like to point out what goes wrong if one applies the idea of the proof to Theorem 2 directly to obtain a faithful discrete representation of \mathbb{F}_2 in $Diff_+^1[0, 1]$:

Let $A_n, B_n, n \in \mathbb{Z}$ be mutually disjoint open subintervals in $(0, 1)$ satisfying conditions (i), (ii) and (iii).

We will show that it is impossible to have the maps differentiable (C^1 class) under these conditions (i)-(iii); there are obstructions easily obtained from the Mean Value Theorem.

Without loss of generality we may assume that A_n, B_n converge to 1 as $n \rightarrow \infty$. Let $\lim_{x \rightarrow 1^-} f'(x) = p$. (Then $p > 0$).

Let p_1, p_2 be positive real numbers such that

$$p_1 < p < p_2, \quad p_1 > \frac{99}{100}p, \quad p_2 < \frac{101}{100}p$$

Then, by the Mean Value Theorem, from condition (ii) we obtain that

$$|B_n| > (p_1 + p_1^2 + p_1^3)|A_n| \text{ and } |B_n| > \left(\frac{1}{p_2} + \frac{1}{p_2^2} + \frac{1}{p_2^3}\right)|A_{n+1}|$$

for sufficiently big positive n . Then

$$\frac{|g(B_n)|}{|B_n|} \leq \frac{|A_{n+1}|}{|B_n|} < \frac{1}{\frac{1}{p_2} + \frac{1}{p_2^2} + \frac{1}{p_2^3}}$$

and

$$\frac{|g^{-1}(B_n)|}{|B_n|} \leq \frac{|A_n|}{|B_n|} < \frac{1}{p_1 + p_1^2 + p_1^3}$$

Then, by the Mean Value Theorem, we obtain that for sufficiently big positive n , there exists $u_n, v_n \in B_n$ such that $g'(u_n) < \frac{1}{\frac{1}{p_2} + \frac{1}{p_2^2} + \frac{1}{p_2^3}}$, and $(g^{-1})'(v_n) < \frac{1}{p_1 + p_1^2 + p_1^3}$.

However, since $\lim_{x \rightarrow 1^-} g'(x) = 1 / \lim_{x \rightarrow 1^-} (g^{-1})'(x)$, we obtain a contradiction because

$$\frac{1}{\frac{1}{p_2} + \frac{1}{p_2^2} + \frac{1}{p_2^3}} \frac{1}{p_1 + p_1^2 + p_1^3} < \frac{1}{\frac{p_1}{p_2} + \frac{p_1^2}{p_2^2} + \frac{p_1^3}{p_2^3}} < \frac{1}{2} < 1$$

Remark 4. In the proof of Theorem 2, by slightly changing conditions (1)-(2), it is possible to replace condition (ii) by the following weaker version:

(ii)' for all $i \in \{1, 2\}, n \in \mathbb{Z}$, $f^i(A_n) \subseteq B_n, f^{-i}(A_n) \subseteq B_{n-1}$

However, a similar argument shows that there are no $f, g \in \text{Diff}_+^1[0, 1]$ satisfying conditions (i), (ii)' and (iii). It also follows from the criterion of D.Calegari in [C] that no C^1 -class diffeomorphisms exist which satisfy conditions (i), (ii)' and (iii).

Remark 5. The metric in $C^1[0, 1]$ is given by the norm $\|f\| = \|f\|_0 + \|f\|_1$ where $\|f\|_0 = \sup_{x \in [0, 1]} |f(x)|$, $\|f\|_1 = \sup_{x \in [0, 1]} |f'(x)|$. If $\|f\|_1$ is small and $f(0) = 0$, then by Mean Value Theorem $\|f\|_0$ cannot be big. However, $\|f\|_1$ can be big even if $\|f\|_0$ is small. In the proof of Theorem 1, taking $f(x) = W(x) - x$, we actually show that $\|f\|_1$ stays big for all $W \neq 1$; we do not show that $\|f\|_0$ is big. However, in the proof of Theorem 2, we indeed show a stronger fact that $\|f\|_0$ remains big.

QUESTIONS

In this section, we raise several questions. We will address these questions in our next publication.

The regularity of the representation is a very interesting question; if a finitely generated group Γ admits a faithful discrete representation in $\text{Diff}_+^1[0, 1]$ or in $\text{Homeo}_+[0, 1]$, it is interesting to know if one can achieve faithful discrete representations of higher (C^k , $k > 1$, C^∞ , analytic, etc) regularity.

Question 1. Does a free group \mathbb{F}_2 admit a faithful discrete representation into $\text{Diff}_+^1[0, 1]$

- a) of C^k regularity for some $k > 1$?
- b) of C^k regularity for any $k \geq 1$?
- c) of C^∞ regularity?
- d) of analytic regularity?

Let Γ be a finitely generated group, and $\pi : \Gamma \rightarrow \text{Diff}_+^1[0, 1]$ be a faithful discrete representation of it.

Definition 4. The representation π is called $\|\cdot\|_0$ -discrete if there exists $C > 0$ such that $\|\pi(g)\|_0 > C$ for all $g \in \Gamma \setminus \{1\}$.

By Remark 5, $\|\cdot\|_0$ -discreteness of the representation implies its discreteness in $\text{Diff}_+^1[0, 1]$. Also, a $\|\cdot\|_0$ -discrete representation of a group into $\text{Diff}_+^1[0, 1]$ is just a discrete representation into $\text{Homeo}_+[0, 1]$ of C^1 -regularity.

Question 2. Does \mathbb{F}_2 admit a faithful $\|\cdot\|_0$ -discrete representation into $Diff_+^1[0, 1]$?

Definition 5. The representation π is called *strongly discrete* if there exists $C > 0$ and $x_0 \in (0, 1)$ such that $|\pi(g)'(x_0) - 1| > C$ for all $g \in \Gamma \setminus \{1\}$.

Question 3. Does \mathbb{F}_2 admit a faithful strongly discrete representation into $Diff_+^1[0, 1]$?

Similarly, we say that a faithful representation $\pi : \Gamma \rightarrow Homeo_+[0, 1]$ is *strongly discrete* (in $Homeo_+[0, 1]$) if there exists $C > 0$ and $x_0 \in (0, 1)$ such that $|\pi(g)(x_0) - x_0| > C$ for all $g \in \Gamma \setminus \{1\}$. Notice that in the proof of Theorem 2, the representation of \mathbb{F}_2 into $Homeo_+[0, 1]$ is indeed strongly discrete.

Definition 6. Let G be a topological group or a group with a metric. We say that the *Weak Margulis Lemma* holds for G , if there exists an open non-empty neighborhood U of identity such that any discrete subgroup of G generated by elements from U does not contain a non-abelian free subgroup.

We will be interested in the group $Diff_+^{1+\epsilon}[0, 1]$ where ϵ is a fixed positive real number. On this group, we are considering the metric d_1 , i.e. the metric which comes from the Banach norm of $C^1[0, 1]$

Question 4. Does the Weak Margulis Lemma hold for the group $Diff_+^{1+\epsilon}[0, 1]$ for some $\epsilon > 0$?

Remark 6. It follows from the proof of Theorem 1 and from Theorem 2 that the Weak Margulis Lemma does not hold neither for $Diff_+^1[0, 1]$ nor for $Homeo_+[0, 1]$, in respective metrics.

The study of discrete subgroups of $Diff_+^1[0, 1]$ is interesting beyond the existence question of discrete faithful representations of free groups or even of the groups which contain non-abelian free subgroups. The existence of a faithful representation into $Diff_+^1[0, 1]$ imposes some algebraic properties onto the group; for example, it is well-known that any subgroup of $Homeo_+[0, 1]$ is left-orderable ([**G**]). Furthermore, if a group is isomorphic to a subgroup of $Diff_+^1[0, 1]$ then it is locally indicable, as proven by W.Thurston ([**Th**]). It is interesting to consider if discreteness implies further algebraic restrictions on the group. We would like to ask the following

Question 5. Is there a finitely generated group which admits a faithful representation into $\text{Diff}_+^1[0, 1]$ but does not admit a faithful discrete representation?

Acknowledgment: I am thankful to Matthew G. Brin and Danny Calegari for useful discussions related to the content of this paper. I also would like to thank to two anonymous referees for valuable comments.

R e f e r e n c e s :

[B] Bergman, M. G. Right orderable groups that are not locally indicable. *Pacific Journal of Mathematics*. **147** Number 2 (1991), 243-248.

[C] Calegari, D. Nonsmoothable, locally indicable group actions on the interval. *Algebraic and Geometric Topology* **8** (2008) no. 1, 609-613

[FF1] Farb, B., Franks, J. Groups of homeomorphisms of one-manifolds, II: Extension of Holder's Theorem. *Transactions of the American Mathematical Society*. **355**, (2003) , no.11, 4385-4396.

[FF2] Farb, B., Franks, J. Groups of homeomorphisms of one-manifolds, III: Nilpotent subgroups. *Ergodic Theory and Dynamical Systems*. **23**, (2003), no.5, 1467-1484.

[FS] Farb, B., Shalen, P. Groups of real-analytic diffeomorphisms of the circle. *Ergodic Theory and Dynamical Systems*. **22**, (2002), 835-844.

[G] Ghys, E. Groups acting on the circle. *Enseign. Math.* (2) **47**(3-4):329-407, 2001.

[N1] Navas, A. Growth of groups and diffeomorphisms of the interval. *GAF*. **18**, (2008), 988-1028.

[N2] Navas, A. Sur le groupes de difféomorphismes du cercle engendrés par des éléments proches des rotations. *L'Enseignement Mathématique* **50** (2004), 29-68.

[N3] Navas, A. A finitely generated locally indicable group with no faithful action by C^1 diffeomorphisms of the interval. *Geometry and Topology* **14** (2010) 573-584.

[R] Raghunathan, M.S. *Discrete subgroups of semi-simple Lie groups*. Springer-Verlag, New York 1972. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 68.

[Th] Thurston, W. A generalization of Reeb stability theorem. *Topology* **13**, (1974) 347-352.

[Ts] Tsuboi, T. Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle. *Journal of Mathematical Society of Japan*. **47** (1995) 1-30.

[Y] Yoccoz, J.C. Centralisateurs et conjugaison différentiable des difféomorphismes du cercle. Petits diviseurs en dimension 1. *Astérisque* **231** (1995), 89-242.

AZER AKHMEDOV, DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, ND, 58102, USA

E-mail address: `azer.akhmedov@ndsu.edu`