

Some applications of Hölder's theorem in groups of analytic diffeomorphisms of 1-manifolds

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ABSTRACT: We obtain a simple obstruction to embedding groups into the analytic diffeomorphism groups of 1-manifolds. Using this, we classify all RAAGs which embed into $\text{Diff}_+^\omega(\mathbb{S}^1)$. We also prove that a branch group does not embed into $\text{Diff}_+^\omega(\mathbb{S}^1)$.

1. INTRODUCTION

It is a classical fact (essentially due to Hölder, [cf] [9]) that if a subgroup $\Gamma \leq \text{Homeo}_+(\mathbb{R})$ acts freely then Γ is Abelian. This is obtained by first showing that Γ is necessarily left-orderable and Archimedean, and then showing that Archimedean groups are Abelian.

It is interesting to ask the same question for an arbitrary manifold, i.e. if M is an orientable manifold and a group $\Gamma \leq \text{Homeo}_+(M)$ acts freely then is Γ necessarily Abelian? Motivated by this question, we will say that an orientable manifold M is *Hölder* if any freely acting subgroup of $\text{Homeo}_+(M)$ is Abelian. Similarly, if M is an orientable manifold with boundary ∂M , then we will say that M is *Hölder* if any freely acting subgroup $\Gamma \leq \text{Homeo}_+(M, \partial M)$ is Abelian. At the moment we have very little understanding of Hölder manifolds. For example, we do not even know if the 2-dimensional cube I^2 is Hölder.¹ On the other hand, it follows from the Lefschetz fixed-point theorem that even-dimensional spheres \mathbb{S}^{2n} , $n \geq 1$ (and more generally, even-dimensional rational homology spheres) are trivially Hölder. Interestingly, the circle \mathbb{S}^1 is also a (non-trivial) Hölder manifold, that is any freely acting subgroup of orientation preserving circle homeomorphisms is necessarily Abelian (see [9]).

Hölder's theorem provides strong and sometimes surprising obstructions to embedding groups into the homeomorphism group of the circle. For example, it immediately implies that (as remarked in [9]) a finitely generated infinite torsion group does not embed in $\text{Homeo}_+(\mathbb{S}^1)$. Let us emphasize that even for the simplest 2-manifolds such as \mathbb{S}^2 or \mathbb{T}^2

¹This question seems to be related to the question of Calegari and Rolfsen in [5] about left-orderability of $\text{Homeo}_+(I^n, \partial I^n)$, $n \geq 2$. Also, in [4], Calegari proves a striking topological consequence of the freeness but only for \mathbb{Z}^2 -actions, namely, for a free C^1 class \mathbb{Z}^2 action by orientation preserving homeomorphisms of the plane, the Euler class of the action must vanish!

this question is still unsettled; recently, some interesting partial results have been obtained by N.Guelman and I.Liousse (see [7] and [8]).

In this paper we study obstructions for embeddability of groups into the group of analytic diffeomorphisms of 1-manifolds. Throughout the paper, M^1 will denote a compact oriented 1-manifold, so we will assume that either $M^1 = I$ or $M^1 = \mathbb{S}^1$, with a fixed orientation. $\text{Diff}_+^\omega(M^1)$ will denote the group of orientation preserving analytic diffeomorphisms of M^1 .

The following lemma shows that the commutativity relation among the elements of $\text{Diff}_+^\omega(I)$ is transitive.

Lemma 1.1. *Let $f, g, h \in \text{Diff}_+^\omega(I)$ be non-trivial elements such that f commutes with g and g commutes with h . Then f commutes with h .*

Proof. A non-trivial analytic diffeomorphism has finitely many fixed points. Therefore if two non-trivial analytic diffeomorphisms ϕ_1, ϕ_2 commute then $\text{Fix}(\phi_1) = \text{Fix}(\phi_2)$.

Thus we obtain that $\text{Fix}(f) = \text{Fix}(g) = \text{Fix}(h)$. Let us assume that $\text{Fix}(f) = \text{Fix}(g) = \text{Fix}(h) = \{p_1, \dots, p_n\}$, where $0 = p_1 < p_2 < \dots < p_n = 1$.

Now, assume that f and h do not commute, and let H be a subgroup of $\text{Diff}_+^\omega(I)$ generated by f and h . H fixes all the points p_1, \dots, p_n . Then, by Hölder's Theorem, H contains a non-trivial element ϕ such that ϕ has a fixed point $p \in (0, 1)$ distinct from p_1, \dots, p_n . Let p lie in between p_i and p_{i+1} for some $i \in \{1, \dots, n\}$. Since $[\phi, g] = 1$, we obtain that ϕ must have infinitely many fixed points in (p_i, p_{i+1}) . Contradiction. \square

Lemma 1.1 implies strong restrictions for the embedding of groups into $\text{Diff}_+^\omega(M^1)$. We would like to observe the following fact, which will be useful for our consideration of right-angled Artin groups in Section 2.

Lemma 1.2. *Let f_1, \dots, f_N be non-identity elements of $\text{Diff}_+^\omega(I)$. Let also G be a simple graph on N vertices v_1, \dots, v_N such that (v_i, v_j) is an edge whenever f_i and f_j commute. Assume that G is connected. Then G is isomorphic to a complete graph K_N .*

Proof. Let $1 \leq i < j \leq N$. We want to show that v_i and v_j are connected with an edge. Let (u_0, u_1, \dots, u_m) be a path in G such that $u_0 = v_i, u_m = v_j$. If (u_0, u_j) is an edge, and $j < m$, then, by Lemma

1.1, (u_0, u_{j+1}) is an edge. Since (u_0, u_1) is an edge, by induction, we obtain that (u_0, u_m) is an edge. \square

Since Hölder's Theorem holds for the analytic diffeomorphism group of the circle as well, slightly weaker versions of both of the lemmas 1.1 and 1.2 generalize to $\text{Diff}_+^\omega(M^1)$.

Lemma 1.3. *Let $f, g, h \in \text{Diff}_+^\omega(\mathbb{S}^1)$ be elements of infinite order, f commutes with g and g commutes with h . Then f^m commutes with h^m for some $m \geq 1$.*

Proof. Let Γ be a subgroup generated by f and h . If Γ acts freely then by Hölder's Theorem it is Abelian, and the claim follows. If Γ does not act freely then let p be a fixed point for some non-trivial $\gamma \in \Gamma$. By analyticity, γ has finitely many fixed points, and therefore since $[g, \gamma] = 1$, p is a fixed point of g^n for some $n \geq 1$. Then g^n has finitely many fixed points, and $[f, g^n] = [h, g^n] = 1$, so f^i and h^j both fix p for some positive integers i, j . Setting $m = ij$, p is a fixed point of both f^m and h^m . The claim now follows from Lemma 1.1 applied to f^m, g^n , and h^m , since the subgroup of elements of $\text{Diff}_+^\omega(\mathbb{S}^1)$ which fix p may be identified with a subgroup of $\text{Diff}_+^\omega(I)$. \square

Now we obtain an analogue of Lemma 1.2.

Lemma 1.4. *Let f_1, \dots, f_N be elements of $\text{Diff}_+^\omega(\mathbb{S}^1)$ of infinite order. Let also G be a simple graph on N vertices v_1, \dots, v_N such that $(v_i v_j)$ is an edge whenever f_i^m and f_j^m commute for some $m \geq 1$. Assume that G is connected. Then G is isomorphic to a complete graph K_N . \square*

Remark 1.5. It is easy to see that Lemma 1.1 and Lemma 1.3 both fail for the the group of orientation preserving analytic diffeomorphisms of the real line. Indeed, for a real number $a \neq 0$, let $f_a(x) = \frac{1}{2} \sin(ax) + x$ and $g_a(x) = x + \frac{2\pi}{a}$. Then, for all distinct $a, b \in \mathbb{R} \setminus \{0\}$, we have $[f_a, g_a] = 1$ and $[g_a, g_b] = 1$ while for some a, b , the diffeomorphisms f_a^n, f_b^n do not commute for any $n \geq 1$. However, it is possible to prove a similar but much weaker statement for $\text{Diff}_+^\omega(\mathbb{R})$, and then an analogue of Lemma 1.4, at the expense of increased technicality. We skip the relevant discussion here.

2. CLASSIFICATION OF RAAGS IN $\text{Diff}_+^\omega(M^1)$

A right angled Artin group (RAAG) is defined as follows. Let $G = (V, E)$ be a finite simple graph with a set of vertices $V = \{v_1, \dots, v_n\}$,

and Γ be a finitely presented group given by the presentation

$$\Gamma = \langle x_1, \dots, x_n \mid [x_i, x_j] = 1 \text{ iff } (v_i v_j) \text{ is an edge in } G \rangle.$$

Thus the graph G defines the group Γ . RAAGs have been studied extensively in the past decades from combinatorial, algebraic and geometric points of view. In the recent paper [1] the authors prove the following very interesting result.

Theorem 2.1. *Every RAAG embeds in $\text{Diff}_+^\infty(\mathbb{R})$.*

The question of whether or not every RAAG embeds in $\text{Diff}_+^\infty(M^1)$ is left open in [1]. The following theorem gives a necessary and sufficient condition under which a RAAG embeds in $\text{Diff}_+^\omega(M^1)$.

Theorem 2.2. *A RAAG $\Gamma = \Gamma(G)$ embeds in $\text{Diff}_+^\omega(M^1)$ if and only if every connected component of the graph G is a complete graph.*

Proof. We will present a constructive proof. The “only if” part follows immediately from Lemmas 1.2 and 1.3 for $M^1 = I$ and $M^1 = \mathbb{S}^1$, respectively. For the “if” part, let C_1, \dots, C_m be all connected components of G , such that C_i is a complete graph on m_i vertices.

We let $f, g_n, n \geq 1$ be non-trivial orientation preserving analytic diffeomorphisms of M^1 satisfying the following conditions:

(c1) for every finite subset $A \subseteq \mathbb{N}$ the diffeomorphisms $g_n, n \in A$ generate a free Abelian group of rank $|A|$.

(c2) the subgroup generated f and $g_n, n \geq 1$ is isomorphic to the free product $\langle f \rangle * \langle g_1, g_2, \dots \rangle$.

Now, let $h_{i,n} = f^i g_n f^{-i}$ for all non-negative i, n . For all $i \in \{1, \dots, m\}$, let H_i be a subgroup generated by $h_{i,1}, \dots, h_{i,m_i}$.

Notice that, for all $i \in \{1, \dots, m\}$, the group H_i is a free Abelian group of rank m_i . Let Γ be a subgroup generated by H_1, \dots, H_m . Then $\Gamma = H_1 * H_2 * \dots * H_m$ thus Γ is isomorphic to the RAAG of the graph G .

It remains to show that there exist elements $f, g_n, n \geq 1$ satisfying conditions (c1) and (c2). Let $\alpha_1, \alpha_2, \dots$ be rationally independent numbers in $(1, \pi)$ such that $\log \alpha_1, \log \alpha_2, \dots$ are also rationally independent. In the case of $M^1 = I$, we choose $g_n(x) = \frac{\alpha_n x}{(\alpha_n - 1)x + 1}$ for all $n \geq 1$, and in the case of $M^1 = \mathbb{S}^1$, we choose g_n to be the orientation preserving rotation by the angle α_n .

Then $g_n, n \geq 0$ satisfy condition (c1). Now, we need to prove that there exists $f \in \text{Diff}_+^\omega(M^1)$ such that condition (c2) holds.

We indicate the construction for $M^1 = I$. Let F be a free group formally generated by letters f, g_1, g_2, \dots (so F is a free group with infinite rank; also, we abuse the notation by denoting the generators of F by g_1, g_2, \dots which are already defined analytic diffeomorphisms, and by f which is an analytic diffeomorphism we intend to define). Let $A = \{f, f^{-1}, g_1, g_1^{-1}, g_2, g_2^{-1}, \dots\}$, and W_1, W_2, \dots be *all* non-trivial reduced words in the alphabet A such that each W_i contains f or f^{-1} .

Let D be an open connected and bounded domain in \mathbb{C} such that D contains the real interval $[0, 1]$. We build f as a sum $f = \sum_{n=0}^{\infty} \omega_n$ of analytic functions ω_n on D . Intuitively speaking, each n -th summand ω_n induces a small perturbation which prevents the corresponding n -th word $W_n(f, g_1, g_2, \dots)$ from reducing to identity. We recursively define the maps $\omega_0, \omega_1, \omega_2, \dots$, as well as sequences D_1, D_2, \dots and $\epsilon_1, \epsilon_2, \dots$ of real numbers as follows.

Let ω_0 be an identity map, i.e. $\omega_0(z) = z, \forall z \in D$. Fix an arbitrary point $p \in (0, 1)$. We let $\omega_1 : D \rightarrow \mathbb{C}$ be an analytic function such that

- (i) $\omega_1(0) = \omega_1(1) = 0$,
- (ii) $\omega_1(x)$ is real for all $x \in \mathbb{R} \cap D$,
- (iii) $\sup_{z \in D} |\omega_1(z)| < \frac{1}{4}$,
- (iv) $\sup_{x \in [0, 1]} |\omega_1'(x)| < \frac{1}{4}$, and
- (v) $W_1(f_1, g_1, g_2, \dots)(p) \neq p$, where $f_1 = \omega_0 + \omega_1$.

Let $D_1 = |W_1(f_1, g_1, g_2, \dots)(p) - p|$. Let $\epsilon_1 < 1$ be a positive number so small that if $\sup_{x \in [0, 1]} |\varphi(x) - f_1(x)| < \epsilon_1$, then

$$\sup_{x \in [0, 1]} |W_1(\varphi, g_1, g_2, \dots)(x) - W_1(f_1, g_1, g_2, \dots)(x)| < \frac{D_1}{2}.$$

Suppose now $m \geq 2$ and $\omega_1, \dots, \omega_{m-1}$ are chosen so that the quantity $D_{m-1} = |W_{m-1}(f_{m-1}, g_1, g_2, \dots)(p) - p|$ is strictly positive where $f_{m-1} = \omega_0 + \omega_1 + \dots + \omega_{m-1}$. Also suppose $\epsilon_1, \dots, \epsilon_{m-1}$ are chosen so that $\epsilon_{m-1} \leq \dots \leq \epsilon_1 \leq 1$, and if $\sup_{x \in [0, 1]} |\varphi(x) - f_{m-1}(x)| < \epsilon_{m-1}$, then

$$\sup_{x \in [0, 1]} |W_i(\varphi, g_1, g_2, \dots)(x) - W_i(f_{m-1}, g_1, g_2, \dots)(x)| < \frac{D_i}{2}$$

for each $1 \leq i \leq m-1$. Then choose $\omega_m : D \rightarrow \mathbb{C}$ such that:

- (i) $\omega_m(0) = \omega_m(1) = 0$,
- (ii) $\omega_m(x)$ is real for all $x \in \mathbb{R} \cap D$,

- (iii) $\sup_{z \in D} |\omega_m(z)| < \frac{\epsilon_{m-1}}{2^{m+1}}$,
- (iv) $\sup_{x \in [0,1]} |\omega'_m(x)| < \frac{1}{2^{m+1}}$, and
- (v) $W_m(f_m, g_1, g_2, \dots)(p) \neq p$, where $f_m = \omega_0 + \omega_1 + \dots + \omega_m$.

Let $f(z) = \sum_{n=0}^{\infty} \omega_n(z)$ for $z \in D$. By condition (iii) f is a uniform limit of analytic functions on an open bounded domain and hence analytic.

Moreover, $f'(x)$ is real for all $x \in [0, 1]$, and $f'(x) = \omega'_0(x) + \sum_{n=1}^{\infty} \omega'_n(x) \geq$

$1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} > 0$ by condition (iv), so f is increasing and hence an analytic diffeomorphism of I since it fixes 0 and 1. For every word W_n , we have $W_n(f, g_1, g_2, \dots)(p) \neq p$, since $\sup_{x \in [0,1]} |f(x) - f_n(x)| \leq$

$$\sum_{i=n+1}^{\infty} \frac{\epsilon_i}{2^{i+2}} \leq \frac{\epsilon_n}{2} < \epsilon_n \text{ and therefore}$$

$$\begin{aligned} |W_n(f, g_1, g_2, \dots)(p) - p| &\geq |W_n(f_n, g_1, g_2, \dots)(p) - p| - |W_n(f, g_1, g_2, \dots)(p) \\ &\quad - W_n(f_n, g_1, g_2, \dots)(p)| \\ &> D_n - \frac{D_n}{2} = \frac{D_n}{2}. \end{aligned}$$

Thus condition (c2) is satisfied. (Obtaining an analytic diffeomorphism of the circle satisfying (c2) may be done similarly.) \square

3. MORE APPLICATIONS

In this section we will observe other consequences of Lemma 1.2 and Lemma 1.4.

Proposition 3.1. *Let Γ be a group with a sequence of elements g_1, g_2, \dots such that the centralizer C_{g_i} is a proper subset of the centralizer $C_{g_{i+1}}$ for every $i \geq 1$. Then Γ does not embed in $\text{Diff}_+^{\omega}(I)$.*

Proof. We will prove a much stronger fact. By the assumption, in Γ , there exist elements g_1, g_2, h_1, h_2 such that h_1 belongs to the centralizer of g_1 , h_1, h_2 belong to the centralizer of g_2 while h_2 does not belong to the centralizer of g_1 .

Let $V = \{g_1, g_2, h_1, h_2\}$. Consider a simple graph $G = (V, E)$ with a vertex set V and where the edge set E defined as follows: $(v_i v_j) \in E$ iff v_i and v_j are distinct and $[v_i, v_j] = 1$ in the group Γ .

Notice that $(g_1, h_1), (g_2, h_1), (g_2, h_2)$ are edges in G thus G is connected. By Lemma 1.2 G is isomorphic to K_4 . Hence $[g_1, h_2] = 1$. Contradiction. \square

The existence of strictly increasing chain of centralizers is an interesting property for groups; most notably in the area of branch groups.

Proposition 3.2. *A branch group does not embed in $\text{Diff}_+^\omega(M^1)$.*

Proof. a) For $M^1 = I$, this is immediate from Lemma 1.1. Indeed, by definition of a branch group (see [2]), it contains elements f, g, h where $[f, g] = [g, h] = 1$ while $[f, h] \neq 1$.

b) For $M^1 = \mathbb{S}^1$, let $\Gamma \leq \text{Diff}_+^\omega(M^1)$ be a branch group. Then, by definition of a branch group, Γ has a finite index subgroup of the form $\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ where $n \geq 2$ and none of the groups $\Gamma_i, 1 \leq i \leq n$ is virtually Abelian. By Hölder's Theorem for the circle, there exists a non-trivial diffeomorphism $f \in \Gamma_n$ such that $\text{Fix}(f) \neq \emptyset$. Let p_1, \dots, p_N be all fixed points of f listed along the orientation of the circle \mathbb{S}^1 .

Let also $H = \{\gamma \in \Gamma_1 \mid \text{Fix}(\gamma) \neq \emptyset\}$. Then for every $\gamma \in H \setminus \{1\}$, by analyticity, we have $\text{Fix}(\gamma) = \text{Fix}(f)$. On the other hand, if $\gamma(p_i) = p_j$ for some $\gamma \in \Gamma_1$ and $i, j \in \{1, \dots, N\}$, then $\gamma(p_{i+1}) = p_{j+1}$. Hence H is a finite index subgroup of Γ_1 . Also, for every $h_1, h_2 \in H$, we have $[h_1, f] = [h_2, f] = 1$. Then, by Lemma 1.1, $[h_1, h_2] = 1$. Thus H is Abelian. But then Γ_1 is virtually Abelian. Contradiction. \square

Now, we would like to observe several other corollaries of the results from Section 1.

Corollary 3.3. *A non-Abelian group with a non-trivial center does not embed in $\text{Diff}_+^\omega(I)$.*

This fact (with modified statement) can be deduced already for the group $\text{Diff}_+^2(I)$ using Kopell's Lemma. More precisely, it follows from Kopell's Lemma that an irreducible subgroup of $\text{Diff}_+^2(I)$ has a trivial center.

Notice that $SL(n, \mathbb{Z}), n \geq 3$ contains a copy of the integral 3×3 unipotent subgroup. Then, by Margulis arithmeticity result of higher rank lattices, and by Corollary 3.3 and Lemma 1.3, we obtain the following corollary.

Corollary 3.4. A lattice in $SL(n, \mathbb{R})$, $n \geq 3$ does not embed in $\text{Diff}_+^\omega(M^1)$.

Let us emphasize that it is already known that a finite index subgroup of $SL(n, \mathbb{Z})$, $n \geq 3$ does not embed in $\text{Homeo}_+(I)$, see [11]; for an arbitrary lattice in $SL(n, \mathbb{R})$, $n \geq 3$ this question is still open. On the other hand, it is also known that a lattice in $SL(n, \mathbb{R})$, $n \geq 3$ does not embed in $\text{Diff}_+(M^1)$ as proved by Ghys and Burger-Monod, and it is also known that any infinite discrete group with property (T) does not embed in $\text{Diff}_+^{1+\alpha}(\mathbb{S}^1)$, $\alpha > \frac{1}{2}$ as proved by Navas [10]. (in particular, the result of Corollary 3.4 is not new.)

Remark 3.5. The only property of non-trivial analytic diffeomorphisms of M^1 that we use in the paper is the finiteness of the fixed point set. Thus all the results of this paper holds for any subgroup $\Gamma \leq \text{Homeo}_+(M^1)$ where every non-identity element of Γ has finitely many fixed points.

It is interesting to study representations of general Artin groups into the group of analytic diffeomorphisms of manifolds. Notice that, by Corollary 3.3 and Lemma 1.3, the braid groups on $n \geq 3$ strings (another special subclass of Artin groups) do not embed in $\text{Diff}_+^\omega(M^1)$.

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