

## Groups not acting on compact metric spaces by homeomorphisms

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ABSTRACT. We show that the direct sum of uncountably many non-Abelian groups does not embed into the group of homeomorphisms of a compact metric space.

The main result of this note does not seem to exist in the literature. Among other things, it provides simple examples of left-orderable groups of continuum cardinality which do not embed in  $\text{Homeo}_+(\mathbb{R})$ .<sup>1</sup>

**Theorem 1.1.** *Let  $X$  be a compact metric space,  $I$  be an uncountable set, and  $G_\alpha$  be a non-Abelian group, for all  $\alpha \in I$ . Then the direct sum*

$$\bigoplus_{\alpha \in I} G_\alpha$$

*does not embed into the group of homeomorphisms of  $X$ .*

**Proof.** We will assume that the direct sum

$$\bigoplus_{\alpha \in I} G_\alpha$$

embeds in  $\text{Homeo}(X)$ .

For all  $\alpha \in I$ , let  $f_\alpha, g_\alpha$  be some noncommuting elements in  $G_\alpha$ . We will denote the metric on  $X$  by  $d(\cdot, \cdot)$ . Notice that the group  $\text{Homeo}(X)$  of homeomorphisms of  $X$  also becomes a metric space via the metric

$$D(\phi, \psi) = \sup_{x \in X} d(\phi(x), \psi(x))$$

for  $\phi, \psi \in \text{Homeo}(X)$ .

By compactness of  $X$ , we have a sequence  $C_n, n \geq 1$  of finite subsets of  $X$  such that:

- (i) For all  $n \geq 1$ ,  $C_n$  is a  $\frac{1}{n}$ -net, i.e., for all  $x \in X$ , there exists  $z \in C_n$  such that  $d(z, x) < \frac{1}{n}$ .

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<sup>1</sup>It is well known that a countable group is left-orderable iff it embeds into  $\text{Homeo}_+(\mathbb{R})$ . Thus it becomes an interesting question whether or not there exists a left-orderable group of continuum cardinality which does not embed in  $\text{Homeo}_+(\mathbb{R})$ . This question has been addressed in a very recent paper [2] where two different examples have been constructed.

- (ii)  $C_n \subseteq C_{n+1}$ , for all  $n \geq 1$ .
- (iii)  $\bigcup_{n \geq 1} C_n = X$ .

Enumerate  $C_n = \{x_1^{(n)}, \dots, x_{p_n}^{(n)}\}$ ,  $n \geq 1$ . For all  $n \geq 1$  and  $\delta > 0$ , we will also write

$$\mathcal{F}_{n,\delta} = \{\phi \in \text{Homeo}(X) \mid \text{diam}\phi(B_{\frac{\delta}{2}}(x_i^{(n)})) < \delta, \forall i \in \{1, \dots, p_n\}\}$$

Since

$$\text{Homeo}(X) \setminus \{1\} = \bigcup_{i \geq 1} \left\{ \phi \mid D(\phi, 1) > \frac{1}{i} \right\}$$

there exists  $c > 0$  and an uncountable set  $I_1 \subset I$  such that for all  $\alpha \in I_1$  we have  $D([f_\alpha, g_\alpha], 1) > c$ .

By compactness, all homeomorphisms of  $X$  are uniformly continuous. Then there exists an uncountable set  $I_2 \subset I_1$  and positive integers  $n, m$  such that  $\max\{\frac{1}{n}, \frac{1}{m}\} < \frac{c}{10}$  and for all  $\beta \in I_2$  we have  $\{f_\beta, g_\beta\} \subseteq \mathcal{F}_{n, \frac{1}{m}}$ .

Since  $\text{Homeo}(X)$  is separable, there exists an uncountable set  $I_3 \subset I_2$  and  $f_\star, g_\star \in \text{Homeo}(X)$  such that for all  $\gamma \in I_3$  we have

$$\max\{D(f_\gamma, f_\star), D(g_\gamma, g_\star)\} < \frac{1}{100n}.$$

Then for all  $\gamma, \eta \in I_3$  and for all  $x \in X$  we have the inequalities

$$d(g_\eta f_\gamma(x), g_\gamma f_\gamma(x)) < \frac{1}{50n} \quad \text{and} \quad d(f_\gamma g_\eta(x), f_\gamma g_\gamma(x)) < \frac{c}{5}.$$

On the other hand, for any two distinct  $\eta, \gamma \in I_3$  there exists  $x_0 \in X$  such that  $d(f_\gamma g_\gamma(x_0), g_\gamma f_\gamma(x_0)) > c$ .

By the triangle inequality, we obtain that  $d(g_\eta f_\gamma(x_0), f_\gamma g_\eta(x_0)) > \frac{c}{2}$ . Thus  $[g_\eta, f_\gamma] \neq 1$ . Contradiction.  $\square$

**Remark 1.2.** To obtain interesting applications of the main theorem,  $X$  can be taken to be an arbitrary compact manifold (with boundary). In the case of a closed interval  $\mathbb{I} = [0, 1]$ , we obtain a nonembeddability result into  $\text{Homeo}_+(\mathbb{R})$ , i.e., the direct sum of uncountably many isomorphic copies of a non-Abelian group  $G$  does not embed in  $\text{Homeo}_+(\mathbb{R})$ . The result of Theorem 1.1 can be extended to a much larger class of topological spaces. One can also drop or replace the non-Abelian condition on the group  $G$  in certain special contexts.

**Remark 1.3.** It was pointed out by K. Mann that the proof of Theorem 1.1 generalizes: one can replace  $\text{Homeo}(X)$  with any second countable Hausdorff topological group.

**Remark 1.4.** By examining the proof of Theorem 1.1, one can see that we use only the fact that  $\text{Homeo}(X)$  is a first countable,  $T_1$ , and separable topological group. For an arbitrary topological space,

“first countable +  $T_1$  + separable”

is weaker than

“second countable + Hausdorff”

but for topological groups these conditions are equivalent. Indeed, for topological groups, even  $T_0$  implies  $T_{3.5}$  (“Kolmogorov” implies “Tychonov”). Thus  $T_1$  implies  $T_2$ . On the other hand, by the Birkhoff–Kakutani Theorem, “first countable + Hausdorff” implies metrizability. In the presence of metric, separability implies second countability.

In light of Theorem 1.1, the property of (not) admitting a direct sum of uncountably many non-Abelian groups seems interesting beyond the class of second countable groups. Especially, for topological groups or for groups of some topological/analytical content, if one imposes conditions related to separability or if a group resembles separable groups in some way, then it is often meaningful to ask whether or not the given group admits a direct sum of uncountably many non-Abelian groups. In the remaining part of the paper, we would like to discuss two key examples:  $\mathcal{B}(X)$  — the group of invertible linear bounded operators on a separable infinite-dimensional Banach space  $X$ , and  $\mathcal{G}_{+\infty}(\mathbb{R})$  — the group of germs of homeomorphisms of the real line at  $+\infty$ .

For a separable infinite dimensional Banach space  $X$ ,  $\mathcal{B}(X)$  admits many interesting topologies (these topologies do not necessarily make  $\mathcal{B}(X)$  a topological group), thus  $\mathcal{B}(X)$  may seem as a candidate for our property in question. However, the following proposition provides a negative answer.

**Proposition 1.5.** *For a separable infinite-dimensional Banach space  $X$ ,  $\mathcal{B}(X)$  admits a direct sum of uncountably many non-Abelian groups.*

**Proof.** As a Banach space,  $X$  admits a Hamel basis, and by the result of G. W. Mackey [1], the cardinality of a Hamel basis is at least continuum. Let  $\{x_\alpha, y_\alpha\}_{\alpha \in I}$  be a Hamel basis for  $X$ , and  $V_\alpha = \text{Span}(x_\alpha, y_\alpha)$  for all  $\alpha \in I$ . In a Banach space, every finite-dimensional subspace is a complemented subspace. Then  $\mathcal{B}(X)$  contains the direct sum

$$\bigoplus_{\alpha \in I} \mathcal{B}(V_\alpha).$$

Since  $\mathcal{B}(V_\alpha)$  is isomorphic to  $\text{GL}(2, \mathbb{R})$  for each  $\alpha \in I$ , it contains a non-Abelian subgroup.  $\square$

The main theorem of [2] proves that  $\mathcal{G}_{+\infty}(\mathbb{R})$  does not admit an embedding into  $\text{Homeo}_+(\mathbb{R})$ . The next proposition shows that this result follows from our Theorem 1.1 (we prefer to work with the notation  $\mathcal{G}_0(\mathbb{I})$  instead of  $\mathcal{G}_{+\infty}(\mathbb{R})$ ).

**Proposition 1.6.** *The group  $\mathcal{G}_0(\mathbb{I})$  of germs of homeomorphisms at 0 admits a direct sum of uncountably many non-Abelian groups.*

**Proof.** Let  $f_n, g_n \in \text{Homeo}_+(\mathbb{I})$ ,  $n \geq 1$  such that:

- (i) For all  $n \geq 1$ ,  $[f_n, g_n] \neq 1$ .
- (ii) For all distinct  $i, j \geq 1$ ,  $[f_i, f_j] = [f_i, g_j] = [g_i, f_j] = [g_i, g_j] = 1$ .

Let also  $I_k = [a_k, b_k]$ ,  $k \geq 1$  be a sequence of mutually disjoint intervals in  $\mathbb{I}$  such that  $\{a_k\}_{k=1}^{\infty}$  is a decreasing sequence with  $\lim_{k \rightarrow \infty} a_k = 0$ .

We can take “copies” of  $f_n, g_n$  in each  $I_k$  as follows: fix  $\phi_k \in \text{Homeo}_+(I_k, \mathbb{I})$  and define  $f_{k,n} = \phi_k^{-1} f_n \phi_k$  for all  $n, k \geq 1$ .

Now, for all  $\beta \in (1, 2)$  we define a sequence  $s_\beta = \{s_{\beta,k}\}_{k=1}^{\infty}$  of positive integers by letting  $s_{\beta,k} = \lfloor \beta k \rfloor$  for all  $k \geq 1$ . Then for all  $\beta \in (1, 2)$ , we define  $f_\beta, g_\beta \in \text{Homeo}_+(\mathbb{I})$  as follows:

If  $x \in I_k$  for some  $k \geq 1$  then we let

$$f_\beta(x) = f_{k,s(\beta,k)}(x) \quad \text{and} \quad g_\beta(x) = g_{k,s(\beta,k)}(x),$$

however, if  $x \in [0, 1] \setminus \bigcup_{k=1}^{\infty} I_k$  then we let  $f_\beta(x) = g_\beta(x) = x$ .

Notice that if  $\beta_1 < \beta_2$  then (since  $\beta_1 > 1$ )  $s_{\beta_2,n}$  is eventually bigger than  $s_{\beta_1,n}$ ; in particular, there exists  $N = N(\beta_1, \beta_2)$  such that for all  $n > N$  we have  $s_{\beta_1,n} \neq s_{\beta_2,n}$ .

Then for all  $\beta \in (1, 2)$  we have

$$[\overline{f_\beta}, \overline{g_\beta}] \neq 1$$

while for all distinct  $\beta_1, \beta_2 \in (1, 2)$  we have

$$[\overline{f_{\beta_1}}, \overline{f_{\beta_2}}] = [\overline{f_{\beta_1}}, \overline{g_{\beta_2}}] = [\overline{f_{\beta_2}}, \overline{g_{\beta_1}}] = [\overline{g_{\beta_1}}, \overline{g_{\beta_2}}] = 1$$

where for  $h \in \text{Homeo}_+(\mathbb{I})$ ,  $\overline{h}$  denotes the equivalence class of  $h$  in  $\mathcal{G}_0(\mathbb{I})$ .

By letting  $G_\beta$  be the group generated by  $\overline{f_\beta}$  and  $\overline{g_\beta}$ , we obtain that  $\mathcal{G}_0(\mathbb{I})$  contains

$$\bigoplus_{\beta \in (1,2)} G_\beta. \quad \square$$

Motivated by Theorem 1.1 (and applications), we would like to introduce the following property for groups.

**Definition 1.7.** Let  $W(x, y)$  be a nontrivial reduced word in the non-Abelian free group generated by  $x$  and  $y$ . We say a group  $\Gamma$  satisfies property  $(P_W)$  if there exists uncountably many distinct pairs  $(f_\alpha, g_\alpha)_{\alpha \in I}$  of elements of  $\Gamma$  such that the following conditions hold:

- (i)  $W(f_\alpha, g_\alpha) \neq 1$  for all  $\alpha \in I$ .
- (ii)  $W(f_\alpha, g_\beta) = 1$  for all distinct  $\alpha, \beta \in I$ .

When  $W = [x, y]$  then we obtain a property that we have been considering — groups which admit a direct sum of uncountably many non-Abelian groups satisfy this property. If only one letter occurs in  $W(x, y)$ , i.e.,  $W(x, y) = x^n$  or  $W(x, y) = y^n$  for some nonzero integer  $n$ , then it is immediate to see that the property  $(P_W)$  is vacuous; no group satisfies such a property. It is also vacuous for a word like  $W(x, y) = x^{-1}y$ . However,

these are degenerate cases, and the property is indeed very interesting for many (most) other types of words.

It is straightforward to generalize the proof of Theorem 1.1 to obtain the following

**Theorem 1.8.** *Let  $\Gamma$  be a group with a property  $(P_W)$  where  $W(x, y)$  is a nontrivial reduced word in the non-Abelian free group generated by  $x$  and  $y$ . Then  $\Gamma$  does not embed in any second countable Hausdorff topological group.*

Theorem 1.8 has numerous interesting applications. We would like to point out only one: in [2], another very nice example of a group (attributed there to C. Rivas) is presented which is left orderable but does not act on a line, namely, the group generated by  $\{a_s \mid s \in \mathbb{R}\}$  with relations

$$a_t a_s a_t^{-1} = a_s^{-1} \text{ iff } t < s.$$

Here, to apply Theorem 1.8, one can take  $f_s = a_s, g_s = a_s$  for all  $s \in \mathbb{R}$  and let  $W(x, y) = xyx^{-1}y$  (curiously, one can even take  $f_s = a_s, g_s = a_s^3$  for all  $s \in \mathbb{R}$ ).

More generally, if  $W(x, y)$  is a nontrivial reduced word then let  $\Gamma(W)$  be the group generated by  $\{a_s, s \in \mathbb{R}\}$  with relations

$$W(a_t, a_s) = 1 \text{ iff } t < s.$$

Let  $\sigma_x(W), \sigma_y(W)$  denote the sum of exponents of  $x$  and  $y$  in  $W$  respectively. By taking  $f_s = a_s, g_s = a_s$  for all  $s \in \mathbb{R}$ , and applying Theorem 1.8, we obtain the following

**Proposition 1.9.** *Let  $W(x, y)$  be a nontrivial reduced word in the free group generated  $x$  and  $y$ , and  $\sigma_x(W) + \sigma_y(W) \neq 0$ . Then the group  $\Gamma(W)$  does not embed in any second countable Hausdorff topological group.*

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## References

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