

On groups of diffeomorphisms of the interval with finitely many fixed points II

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Abstract: In [6], it is proved that any subgroup of $\text{Diff}_+^\omega(I)$ (the group of orientation preserving analytic diffeomorphisms of the interval) is either metaabelian or does not satisfy a law. A stronger question is asked whether or not the Girth Alternative holds for subgroups of $\text{Diff}_+^\omega(I)$. In this paper, we answer this question affirmatively for even a larger class of groups of orientation preserving diffeomorphisms of the interval where every non-identity element has finitely many fixed points. We show that every such group is either affine (in particular, metaabelian) or has infinite girth. The proof is based on sharpening the tools from the earlier work [1].

1. INTRODUCTION

Throughout this paper we will write Φ (resp. Φ^{diff}) to denote the class of subgroups of $\Gamma \leq \text{Homeo}_+(I)$ (resp. $\Gamma \leq \text{Diff}_+(I)$) such that every non-identity element of Γ has finitely many fixed points. An important class of such groups is provided by $\text{Diff}_+^\omega(I)$ - the group of orientation preserving analytic diffeomorphisms of I . Interestingly, not every group in Φ is conjugate (or even isomorphic) to a subgroup of $\text{Diff}_+^\omega(I)$, see [3]. We will consider a natural metric on $\text{Homeo}_+(I)$ induced by the C^0 -metric by letting $\|f\| = \sup_{x \in [0,1]} |f(x) - x|$.

For an integer $N \geq 0$ we will write Φ_N (resp. Φ_N^{diff}) to denote the class of subgroups of $\text{Homeo}_+(I)$ (resp. $\text{Diff}_+(I)$) where every non-identity element has at most N fixed points. It has been proved in [1] that, for $N \geq 2$, any subgroup of Φ_N^{diff} of regularity $C^{1+\epsilon}$ is indeed solvable, moreover, in the regularity C^2 we can claim that it is metaabelian. In [3], we improve these results and give a complete classification of subgroups of Φ_N^{diff} , $N \geq 2$, even at C^1 -regularity. There, it is also shown that the presented classification picture fails in the continuous category, i.e. within the larger class Φ_N .

The main result of this paper is the following

Theorem 1.1. *Let $\Gamma \leq \text{Diff}_+(I)$ such that every non-identity element has finitely many fixed points. Then either Γ is isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})^n$ for some $n \geq 1$, or it has infinite girth.*

Remark 1.2. In particular, we obtain that a subgroup of $\text{Diff}_+(I)$ where every non-identity diffeomorphism has finitely many fixed points, satisfies no law unless it is isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})^n$ for some $n \geq 1$. This also implies a positive answer to Question (v) from [6]. If Γ is *irreducible* (i.e. it does not have a global fixed point in $(0, 1)$), then one can take $n = 1$.

The proof of Theorem 1.1 relies on the study of diffeomorphism groups which act locally transitively. To be precise, we need the following definitions.

Definition 1.3. A subgroup $\Gamma \leq \text{Diff}_+(I)$ is called *locally transitive* if for all $p \in (0, 1)$ and $\epsilon > 0$, there exists $\gamma \in \Gamma$ such that $\|\gamma\| < \epsilon$ and $\gamma(p) \neq p$.

Definition 1.4. A subgroup $\Gamma \leq \text{Diff}_+(I)$ is called *dynamically 1-transitive* if for all $p \in (0, 1)$ and for every non-empty open interval $J \subset (0, 1)$ there exists $\gamma \in \Gamma$ such that $\gamma(p) \in J$.

The notion of dynamical k -transitivity is introduced in [2] where we also make a simple observation that local transitivity implies dynamical 1-transitivity. Notice that if the group is dynamically k -transitive for an arbitrary $k \geq 1$ then it is dense in C^0 -metric. Dynamical k -transitivity for some high values of k (even for $k \geq 2$) is usually very hard, if possible, to achieve, and would be immensely useful (all the results obtained in [2] are based on just establishing the dynamical 1-transitivity). In this paper, we would like to introduce a notion of *weak transitivity* which turns out to be sufficient for our purposes but it is also interesting independently.

Definition 1.5. A subgroup $\Gamma \leq \text{Diff}_+(I)$ is called *weakly k -transitive* if for all $g \in \Gamma$, for all k points $p_1, \dots, p_k \in (0, 1)$ with $p_1 < \dots < p_k$, and for all $\epsilon > 0$, there exist $\gamma \in \Gamma$ such that $g(\gamma(p_i)) \in (\gamma(p_{i-1}), \gamma(p_{i+1}))$, for all $i \in \{1, \dots, k\}$ and $\gamma(p_k) < \epsilon$ where we assume $p_0 = 0, p_{k+1} = 1$. We say Γ is *weakly transitive* if it is weakly k -transitive for all $k \geq 1$.

The proof of the main theorem will follow immediately from the following four propositions which seem interesting to us independently.

Proposition 1.6. *Any irreducible subgroup of Φ^{diff} is either isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})$ or it is locally transitive.*

Proposition 1.7. *Any irreducible subgroup of Φ^{diff} is either isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})$ or it is weakly transitive.*

Proposition 1.8. *Any locally transitive irreducible subgroup of Φ^{diff} is either isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})$ or it has infinite girth.*

Proposition 1.9. *For any $N \geq 1$, any irreducible subgroup of $\text{Diff}_+(I)$ where every non-identity element has at most N fixed points is isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})$.*

Let us point out that, in $C^{1+\epsilon}$ -regularity, Proposition 1.9 is proved in [2] under somewhat weaker conclusion, namely, by replacing the condition “isomorphic to a subgroup of $\text{Aff}_+(\mathbb{R})$ ” with “solvable”, and in C^2 -regularity by replacing it with “metabelian”. Subsequently, another (simpler) argument for this result is given in the analytic category, by A.Navas [6]. In full generality and strength, Proposition 1.9 is proved in [3]. Thus we need to prove only Proposition 1.6, 1.7 and 1.8. The first of these propositions is proved in Section 2, by modifying the main argument from [1]. The second and third propositions are proved in Section 3; in the proof of the third proposition, we follow the main idea from [4].

Notations: For all $f \in \text{Homeo}_+(I)$ we will write $\text{Fix}(f) = \{x \in (0, 1) \mid f(x) = x\}$. The group $\text{Aff}_+(\mathbb{R})$ will denote the group of all orientation preserving affine homeomorphisms of \mathbb{R} , i.e. the maps of the form $f(x) = ax + b$ where $a > 0$. The conjugation of this group by an orientation preserving homeomorphism $\phi : I \rightarrow \mathbb{R}$ to the group of homeomorphisms of the interval I will be denoted by $\text{Aff}_+(I)$ (we will drop the conjugating map ϕ from the notation). By choosing ϕ appropriately, one can also conjugate $\text{Aff}_+(\mathbb{R})$ to a group of diffeomorphisms of the interval as well, and (by fixing ϕ) we will refer to it as the group of affine diffeomorphisms of the interval.

If Γ is a subgroup from the class Φ then one can introduce the following biorder in Γ : for $f, g \in \Gamma$, we let $g < f$ if $g(x) < f(x)$ near zero. If f is a positive element then we will also write $g \ll f$ if $g^n < f$ for every integer n ; we will say that g is *infinitesimal* w.r.t. f . For $f \in \Gamma$, we write $\Gamma_f = \{\gamma \in \Gamma : \gamma \ll f\}$ (so Γ_f consists of diffeomorphisms which are infinitesimal w.r.t. f). Notice that if Γ is finitely generated with a fixed finite symmetric generating set, and f is the biggest generator, then Γ_f is a normal subgroup of Γ , moreover, Γ/Γ_f is Archimedean, hence Abelian, and thus we also see that $[\Gamma, \Gamma] \leq \Gamma_f$.

2. C_0 -DISCRETE SUBGROUPS OF $\text{Diff}_+(I)$: STRENGTHENING THE RESULTS OF [1]

Let us first quote the following theorem from [1].

Theorem 2.1 (Theorem A). *Let $\Gamma \leq \text{Diff}_+(I)$ be a subgroup such that $[\Gamma, \Gamma]$ contains a free semigroup on two generators. Then Γ is not C_0 -discrete, moreover, there exists non-identity elements in $[\Gamma, \Gamma]$ arbitrarily close to the identity in C_0 metric.*

In [2], Theorem A is used to obtain local transitivity results in C^2 -regularity ($C^{1+\epsilon}$ -regularity) for subgroups from Φ_N^{diff} which have derived length at least three (at least $k(\epsilon)$). However, we need to obtain local transitivity results for subgroups which are 1) non-abelian (not necessarily non-metaabelian); 2) from a larger class Φ^{diff} ; 3) and have only C^1 -regularity.

To do this, first we observe that within the class Φ^{diff} , the condition “ Γ contains a free semigroup” by itself implies that “ $[\Gamma, \Gamma]$ is either Abelian or contains a free semigroup”.

Proposition 2.2. *Let Γ be a non-Abelian subgroup from the class Φ^{diff} . Then either Γ is locally transitive or $[\Gamma, \Gamma]$ contains a free semigroup on two generators.*

For the proof of the proposition, we need the following

Definition 2.3. Let $f, g \in \text{Homeo}_+(I)$. We say the pair (f, g) is *crossed* if there exists a non-empty open interval $(a, b) \subset (0, 1)$ such that one of the homeomorphisms fixes a and b but no other point in (a, b) while the other homeomorphism maps either a or b into (a, b) .

It is a well known folklore result that if (f, g) is a crossed pair then the subgroup generated by f and g contains a free semigroup on two generators (see [7]).

Proof of Proposition 2.2. Without loss of generality we may assume that Γ is irreducible.

Let us first assume that Γ is metaabelian, and let N be a nontrivial Abelian normal subgroup of Γ . By Hölder’s Theorem, there exists $f \in \Gamma$ such that $\text{Fix}(f) \neq \emptyset$. On the other hand, by irreducibility of Γ , $\text{Fix}(g) = \emptyset$ for all $g \in N \setminus \{1\}$.

By Lemma 6.2 in [5], N is not discrete. Hence, for all $\epsilon > 0$, there exists $\omega \in N$ such that $\|\omega\| < \epsilon$. This implies that Γ is locally transitive.

Let us now assume that Γ has derived length more than 2 (possibly infinity). Then $\Gamma^{(1)} = [\Gamma, \Gamma]$ is not Abelian. Then by Hölder's Theorem there exists two elements $f, g \in \Gamma^{(1)}$ such that $\text{Fix}(f) \neq \text{Fix}(g)$. We may assume that (by switching f and g if necessary) there exists $p, q \in \text{Fix}(f) \cup \{0, 1\}$ such that $\text{Fix}(f) \cap (p, q) = \emptyset, \text{Fix}(g) \cap (p, q) \neq \emptyset$. Without loss of generality we may also assume that $f(x) > x, \forall x \in (p, q)$ and $g(p) \geq p$. If $g(p) > p$ then f and g form a crossed pair. But if $g(p) = p$ then for sufficiently big n , f and g^n form a crossed pair. \square

To finish the proof of Proposition 1.6 it remains to show the following

Theorem 2.4. *Let $\Gamma \leq \Phi^{\text{diff}}$ be a subgroup containing a free semigroup in two generators, such that $f'(0) = f'(1) = 1$ for all $f \in \Gamma$. Then Γ is locally transitive.*

Proof. The proof is very similar to the proof of Theorem A from [1] with a crucial extra detail.

Without loss of generality, we may assume that Γ is irreducible. Let $p \in (0, 1), \epsilon > 0, M = \sup_{0 \leq x \leq 1} (|f'(x)| + |g'(x)|)$. Let also $N \in \mathbb{N}$ and $\delta > 0$ such that $1/N < \min\{\epsilon, p, 1 - p\}$ and for all $x \in [1 - \delta, 1]$, the inequality $|\phi'(x) - 1| < 1/10$ holds where $\phi \in \{f, g, f^{-1}, g^{-1}\}$.

Let $W = W(f, g)$ be an element of Γ such that

$$(\{f^i W(1/N) \mid -2 \leq i \leq 2\} \cup \{g^i W(1/N) \mid -2 \leq i \leq 2\}) \subset [1 - \delta, 1]$$

and let m be the length of the reduced word W . Let also $x_i = i/N, 0 \leq i \leq N$ and $z = W(1/N)$.

By replacing the pair (f, g) with (f^{-1}, g^{-1}) if necessary, we may assume that $f(z) \geq z$. Then we can define (see the proof of Theorem A in [1] for the details) $\alpha, \beta \in \Gamma$ and a point $z_0 \in (1 - \delta, 1)$ such that the following conditions hold:

- (i) α and β are positive words in f, g of length at most two,
- (ii) $z_0 = z$ or $z_0 = fg(z)$,
- (iii) $z_0 \leq z$,
- (iv) $z_0 \leq \alpha(z_0) \leq \beta\alpha(z_0)$.

Then $\sup_{0 \leq x \leq 1} (|\alpha'(x)| + |\beta'(x)|) \leq M^2$, and the length of the word W in the alphabet $\{\alpha, \beta, \alpha^{-1}, \beta^{-1}\}$ is at most $2m$.

Now, for every $n \in \mathbb{N}$, let

$$\mathbb{S}_n = \{U(\alpha, \beta)\beta\alpha \mid U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length } n.\}$$

$|\mathbb{S}_n| = 2^n$. Applying Lemma 1 from [1] to the pair $\{\alpha, \beta\}$ we obtain that $V(z_0) \geq z_0$ for all $V \in \mathbb{S}_n$.

Now, let $c_k = \frac{1}{100} - \frac{1}{100(k+1)}$ for all $k \geq 1$. Then there exists a sufficiently big n such that the following two conditions are satisfied:

(i) there exist a subset $\mathbb{S}_n^{(1)} \subseteq \mathbb{S}_n$ such that $|\mathbb{S}_n^{(1)}| > (2 - c_1)^n$, and for all $g_1, g_2 \in \mathbb{S}_n^{(1)}$,

$$|g_1 W(x) - g_2 W(x)| < \frac{1}{(1.9)^n}, \forall x \in \{x_i \mid 1 \leq i \leq N - 1\} \cup \{p\}.$$

(ii) $M^{2m+4}(1.1)^n \frac{1}{(1.9)^n} < \epsilon$.

Then from Mean Value Theorem we obtain that

$$|(g_1 W)^{-1}(g_2 W)(x) - x| < 2\epsilon$$

for all $x \in [0, 1]$.¹

If $g_1 W(p) \neq g_2 W(p)$ for some $g_1, g_2 \in \mathbb{S}_n^{(1)}$ then we are done. Otherwise we define the next set

$\mathbb{S}_n^{(2)} \subseteq \{U(\alpha, \beta)gW \mid g \in \mathbb{S}_n^{(1)}, U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length } n.\}$

such that $|\mathbb{S}_n^{(2)}| > (2 - c_2)^n$, and for all $g_1, g_2 \in \mathbb{S}_n^{(2)}$,

$$|g_1 W(x) - g_2 W(x)| < \frac{1}{(1.9)^n}, \forall x \in \{x_i \mid 1 \leq i \leq N - 1\} \cup \{p\}.$$

Again, by Mean Value Theorem, we obtain that

$$|(g_1 W)^{-1}(g_2 W)(x) - x| < 2\epsilon$$

¹to explain this, we borrow the following computation from [1]: let

$$h_1 = g_1 W, h_2 = g_2 W, y_i = W(x_i), z'_i = g_1(y_i), z''_i = g_2(y_i), 1 \leq i \leq N.$$

Then for all $i \in \{1, \dots, N - 1\}$, we have

$$|h_1^{-1} h_2(x_i) - x_i| = |(g_1 W)^{-1}(g_2 W)(x_i) - x_i| =$$

$$|(g_1 W)^{-1}(g_2 W)(x_i) - (g_1 W)^{-1}(g_1 W)(x_i)| = |W^{-1} g_1^{-1} g_2(y_i) - W^{-1} g_1^{-1} g_1(y_i)|$$

$$= |W^{-1} g_1^{-1}(z''_i) - W^{-1} g_1^{-1}(z'_i)|$$

Since $g_1, g_2 \in \mathbb{S}_n$, by the Mean Value Theorem, we have

$$|h_1^{-1} h_2(x_i) - x_i| \leq M^{2m+4}(1.1)^n |z'_1 - z''_1| < M^{2m+4}(1.1)^n \frac{1}{(1.9)^n}$$

Since m is fixed, for sufficiently big n we obtain that $|h_1^{-1} h_2(x_i) - x_i| < \epsilon$. Then $|h_1^{-1} h_2(x) - x| < 2\epsilon$ for all $x \in [0, 1]$.

for all $g_1, g_2 \in \mathbb{S}_n^{(2)}, x \in [0, 1]$. Again, if $g_1W(p) \neq g_2W(p)$ for some $g_1, g_2 \in \mathbb{S}_n^{(2)}$ then we are done, otherwise we continue the process as follows: if the sets $\mathbb{S}_n \supset \mathbb{S}_n^{(1)} \supset \dots \supset \mathbb{S}_n^{(k)}$ are chosen and $g_1W(p) \neq g_2W(p)$ for all $g_1, g_2 \in \mathbb{S}_n^{(k)}$, then we choose

$\mathbb{S}_n^{(k+1)} \subseteq \{U(\alpha, \beta)g \mid g \in \mathbb{S}_n^{(k)}, U(\alpha, \beta) \text{ is a positive word in } \alpha, \beta \text{ of length } n.\}$

such that $|\mathbb{S}_n^{(k+1)}| > (2 - c_{k+1})^n$, and for all $g_1, g_2 \in \mathbb{S}_n^{(k+1)}$,

$$|g_1W(x) - g_2W(x)| < \frac{1}{(1.9)^n}, \forall x \in \{x_i \mid 1 \leq i \leq N - 1\} \cup \{p\}.$$

Since Γ belongs to the class Φ^{diff} the process will stop after finitely many steps, and we will obtain an element ω with norm less than 2ϵ such that $\omega(p) \neq p$. \square

3. WEAK TRANSITIVITY

In this section we prove Proposition 1.7 and then Proposition 1.8.

First, we need the following lemma which follows immediately from the definition of Γ_f .

Lemma 3.1. *Let Γ be a finitely generated irreducible subgroup of the class Φ^{diff} , f be the biggest generator of Γ with at least one fixed point, $z = \min \text{Fix}(f)$, and $\omega(z) > z$ for some $\omega \in \Gamma_f$. Then there exists $\epsilon > 0$ such that if $0 < a < b < \epsilon$, $f^p(a) < b$ for some $p \geq 4$, then, for sufficiently big n , $f^{p-4}(\omega f^{-n} \omega^{-1}(a)) < \omega f^{-n} \omega^{-1}(b)$. \square*

Proof of Proposition 1.7. Let Γ be an irreducible non-affine subgroup of Φ^{diff} , $g \in \Gamma$, and $0 < p_1 < \dots < p_k < 1$. We may assume that Γ is finitely generated. Then, let f be the biggest generator of Γ . Without loss of generality we may also assume that f has at least one fixed point and $\min \text{Fix}(f) < p_1$.

By dynamical 1-transitivity (i.e. Proposition 1.6), there exist elements $\omega_1, \dots, \omega_k \in \Gamma_f$ such that $\min \text{Fix}(\eta_i) \in (p_i, p_{i+1}), 1 \leq i \leq k$ where $\eta_i = \omega_i \dots \omega_1 f \omega_1^{-1} \dots \omega_i^{-1}$.

We can choose $q \geq 4k$ such that $f^q(x) > g(x)$ for all $x \in (0, \frac{1}{2} \min \text{Fix}(f))$. Now, applying Lemma 3.1 inductively, for sufficiently small $\epsilon > 0$ and sufficiently big n , we obtain that

$$f^{4q-4}(\eta_1^{-n}(x)) < \eta_1^{-n}(y), \text{ for all } x, y \in (0, \epsilon) \text{ where } f^{4q}(x) < y$$

$$f^{4q-8}(\eta_2^{-n}(p_1)) < \eta_2^{-n}(p_2),$$

...

$$f^{4q-4i}(\eta_i^{-n}(p_j)) < \eta_i^{-n}(p_{j+1}), 1 \leq j \leq i-1$$

...

$$f^{4q-4k}(\eta_k^{-n}(p_j)) < \eta_k^{-n}(p_{j+1}), 1 \leq j \leq k-1.$$

Thus it suffices to take $\gamma = \eta_k^{-n}$, and we obtain that

$$\gamma(p_{j-1}) < f^q(\gamma(p_j)) < \gamma(p_{j+1}), 1 \leq j \leq k-1$$

where we assume $p_0 = 1$. Then, we have

$$\gamma(p_{j-1}) < g(\gamma(p_j)) < \gamma(p_{j+1}), 1 \leq j \leq k-1.$$

□

Now we are ready to prove Proposition 1.8. Let Γ be a finitely generated irreducible non-affine subgroup of $\text{Homeo}_+(I)$.

Let $S = \{f_1, \dots, f_s\}$ be a generating set of Γ such that $S \cap S^{-1} = \emptyset$ (in particular, S does not contain the identity element), f be the biggest generator of $S \cap S^{-1}$, and let also $m \geq 10s$. By the definition of the order, there exists $\epsilon > 0$ such that $f(x) \geq \xi(x)$ for all $x \in (0, \epsilon)$, $\xi \in S \cap S^{-1}$.

By Proposition 1.9 we can choose $g \in \Gamma$ such that $|\text{Fix}(g)| > 8m$. Since Γ is irreducible by local transitivity and weak transitivity, we can find $h \in \Gamma$ and $\epsilon > 0$ such that the following conditions hold:

- (i) $\gamma(\text{Span}(hgh^{-1})) \subset (0, \epsilon)$ for all $\gamma \in B_{2m}(1)$,
- (ii) if $\text{Fix}(hgh^{-1}) = \{p_1, \dots, p_k\}$ then $p_{i-1} < f^{2m}(p_i) < p_{i+1}$, $2 \leq i \leq k-1$.
- (iii) $\text{Fix}(f_i) \cap \{p_1, \dots, p_k\} = \emptyset$, for all $i \in \{1, \dots, s\}$.

Now, let $\theta = hgh^{-1}$ and

$$S_n = \{\theta, \theta^{mn} f_1 \theta^{mn}, \theta^{2mn} f_2 \theta^{2mn}, \dots, \theta^{smn} f_s \theta^{smn}\}.$$

Let also

$$\delta = \frac{1}{5} \min\{p_{i+1} - p_i \mid 1 \leq i \leq k-1\} \text{ and } V = \bigsqcup_{2 \leq i \leq k-1} (p_i - \delta, p_i + \delta).$$

We now let $x = \frac{p_j + p_{j+1}}{2}$ where $j = \lfloor \frac{k}{2} \rfloor$. Then $x \notin V$, and for a sufficiently big n , by a standard ping-pong argument, for any non-trivial word W in the alphabet S_n of length at most m , we obtain that $W(x) \in V$, hence $W(x) \neq x$.

Since m is arbitrary, we conclude that $\text{girth}(\Gamma) = \infty$.

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