Questions and Remarks on Discrete and Dense Subgroups of Diff(I) [1]

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In recent decades, many remarkable papers have appeared which are devoted to the study of finitely generated subgroups of $\mathrm{Diff}_+([0,1])$ (see [**Be**], [**C**], [**FF1**], [**FF2**], [**FS**], [**G1**], [**G2**], [**N1**], [**N2**], [**Ts**], [**Y**], [**BW**] only for some of the most recent developments). In contrast, discrete subgroups of the group $\mathrm{Diff}_+([0,1])$ are much less studied. Very little is known in this area especially in comparison with the very rich theory of discrete subgroups of Lie groups which has started in the works of F.Klein and H.Poincaré in the 19th century, and has experienced enormous growth in the works of A.Selberg, A.Borel, G.Mostow, G.Margulis and many others in the 20th century. Many questions which are either very easy or have been studied a long time ago for (discrete) subgroups of Lie groups remain open in the context of the infinite-dimensional group $\mathrm{Diff}_+([0,1])$ and its relatives.

In this survey article, we have collected questions about discrete and dense subgroups of $\mathrm{Diff}_+([0,1])$. Most of the questions are motivated by the problems discussed in [**A1**]-[**A4**]. We have found these questions interesting and not trivial, although we believe that many of the questions we are asking are quite approachable.

It is our pleasure to acknowledge the influences by many sources listed in the references. It is also a pleasure to thank to the participants of the *Dynamics Seminar* at North Dakota State University for their motivating questions.

Throughout the paper, I will denote the closed unit interval $[0,1]$. Let also

$\mathrm{Homeo}_+(I)$ be the group of orientation preserving homeomorphisms of I;

$\mathrm{Diff}_+(I)$ be the group of orientation preserving diffeomorphisms of I;

$\mathrm{Diff}_+^{1+\epsilon}(I)$ be the group of orientation preserving diffeomorphisms of I of regularity $C^{1+\epsilon}$ (a function $f : [0,1] \to \mathbb{R}$ belongs to class $C^{1+\epsilon}$ where $0 < \epsilon < 1$ if $f \in C^1$ and $|f'(x) - f'(y)| \leq M|x-y|^{\epsilon}$ for some $M > 0$ and for all $x, y \in I$).

[1] Future updates of this paper will be posted in the author's web page. Comments are welcome.

$\mathrm{Diff}_+^{\omega}(I)$ be the group of orientation preserving analytic diffeomorphisms of I.

We study (discrete) subgroups of these groups by trying to treat them like Lie groups. It turns out (justified by many strong results in the area) that while the groups $\mathrm{Homeo}_+(I)$ and $\mathrm{Diff}_+(I)$ are still quite *amorphous* (we borrow the term from [**CK**]), the subgroups of $\mathrm{Diff}_+^{1+\epsilon}(I)$ start behaving more like subgroups of Lie groups. We limit ourselves here to mentioning only the following bright results:

Theorem 0.1. *a) (Plante-Thurston,* [**PT**]*) Any nilpotent subgroup of* $\mathrm{Diff}_+^2(I)$ *is Abelian.*
b) (Farb-Franks, [**FF2**]*) Any finitely generated torsion-free nilpotent group embeds in* $\mathrm{Diff}_+(I)$*.*

Theorem 0.2. *(Navas,* [**N1**]*) a) Any finitely generated subgroup of* $\mathrm{Diff}_+^{1+\epsilon}(I)$ *of sub-exponential growth is almost nilpotent.*
b) There exists a finitely generated group of intermediate growth which embeds in $\mathrm{Diff}_+(I)$*.*

Both Theorem 0.1. and Theorem 0.2. strongly indicate that while the group $\mathrm{Diff}_+(I)$ still exhibits a great flexibility and richness in terms of the combinatorial properties of its finitely generated subgroups, as soon as we demand a $C^{1+\epsilon}$ regularity, the behavior of the subgroups become much more tame. The fact that higher regularity implies some algebraic rigidity is a phenomenon known even from the older times dating back to at least Koppel [**Kop**] and Thurston [**Th**].

Instead of I, one can study homeomorphism/diffeomorphism groups of the other connected 1-manifolds: $\mathbb{R}$, $\mathbb{S}^1$ and $[0,1)$, but for the questions that we are interested in, the case of I is the hardest and the most interesting. For some of the questions, we briefly sketch their analogue in the context of Lie groups, and provide references.

1. **Free discrete subgroups**

Question 1. Does the free group $\mathbb{F}_2$ admit a faithful C^1-discrete representation in $\mathrm{Diff}_+^{1+\epsilon}(I)$?

It is known that $\mathbb{F}_2$ does admit a faithful C^1-discrete representation in $\mathrm{Diff}_+(I)$ ([**A1**]). Question 1 (secretly important all along) comes into more light in connection with the recent very original approach of E.Shavgulidze to the amenability problem of subgroups of $\mathrm{Diff}_+(I)$ [**Sh**].

A C^0-discrete representation is clearly C^1-discrete (Mean Value Theorem). So one might ask if $\mathbb{F}_2$ admits a faithful C^0-discrete representation in $\mathrm{Diff}_+(I)$. The answer is negative and follows from Theorem A in [**A2**]: **Theorem A.** *Let* Γ *be a subgroup of* $\mathrm{Diff}_+(I)$*, and* $f, g \in [\Gamma, \Gamma]$ *such that* f *and* g *generate a non-abelian free subsemigroup. Then* Γ *is not* C^0*-discrete.*

Containing a free semigroup is an extremely mild condition. Moreover, from practical point of view, it is usually very easy to prove that certain two elements in a group generate a free semigroup. (this may not be the case for free subgroups! For example, it is not easy to prove that the homeomorphisms $F(x) = x + 1, G(x) = x^3$ of $\mathbb{R}$ indeed generate an isomorphic copy of $\mathbb{F}_2$ [**Wh**].)

Both C^1 and C^0 discreteness are very interesting in higher regularities. For C^0-discreteness, the following two theorems from [**A2**] show that groups admitting C^0-discrete representation in higher regularities is a fairly small class.

Theorem B. *Let* Γ *be a* C^0*-discrete subgroup of* $\mathrm{Diff}_+^{1+\epsilon}[0,1]$*. Then* Γ *is solvable with solvability degree at most* $k(\epsilon)$*.*

Theorem C. *If* Γ *is a* C^0*-discrete subgroup of* $\mathrm{Diff}_+^2[0,1]$ *then* Γ *is metaabelian.*

Theorems B-C are obtained by combining Theorem A with the results of Navas [**N1**], Plante-Thurston [**PT**], and Szekeres [**Sz**].

Question 2. What metaabelian subgroups of $\mathrm{Diff}_+^{1+\epsilon}[0,1]$ admit a C^0-discrete representation?

Question 2 can be asked about some specific and popular examples of metaabelian groups as well. One can probably hope for a classification result here.

Question 3. Let $f(x) = \frac{1}{2}x + \frac{1}{2}x^2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}x^2$. The diffeomorphisms $f, g \in \mathrm{Diff}_+(I)$ generate a subgroup isomorphic to $\mathbb{F}_2$ (we will denote it by $\Gamma_{f,g}$). Is this subgroup C^1-discrete?

We are inclined to think that the answer should be negative but we are missing major steps of the proof. Discreteness and denseness can be thought of as opposite properties. The following question indicates how far we indeed are from the understanding of these concepts in the context of $\mathrm{Diff}_+(I)$:

Question 4. Is the subgroup $\Gamma_{f,g}$ C^0-dense in $\mathrm{Diff}_+(I)$?

Since the maps f and g are analytic, the following question becomes very interesting.

Question 5. Does $\mathbb{F}_2$ admit a faithful C^1-discrete representation in $\mathrm{Diff}_+^{\omega}(I)$?

The main idea (inclination) that we use in trying to answer Question 3 is useful also in here. But there are some general (plausible looking) claims that we have not succeeded in proving. As a side remark, notice that the subgroup generated by f and g is not C^0-discrete, by Theorem A.

In contrast with Lie groups, very little study has been done of generic subgroups of $\mathrm{Diff}_+(I)$ (here,"generic" means that the group is generated by elements from an open dense subset). In the following two questions, by "generic subgroup", we mean a subgroup generated by k generic elements where $k > 1$.

Question 6. Is a generic subgroup of $\mathrm{Diff}_+(I)$ C^1-discrete?

For analytic diffeomorphisms, a result of É.Ghys **[G2]** states that if $f, g \in \mathrm{Diff}_+^{\omega}(I)$ are analytic diffeomorphisms sufficiently close to identity in C^0-metric, then a certain sequence $W_k(f, g)$ converges to identity where W_k is a non-trivial word in the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$ representing an element from the k-th subgroup of the lower central series of the group Γ. Thus, in analytic category, at least for the groups generated by generic elements from a sufficiently small neighborhood of identity, the answer to Question 6 is negative (it is easy to show that generic subgroups are free).

For a connected Lie group, random subgroups with sufficiently many generators turn out be dense (see **[BG], [Wi2]** for the precise statements). Can we expect the same in $\mathrm{Diff}_+(I)$? More precisely,

Question 7. Is a generic subgroup of $\mathrm{Diff}_+(I)$ C^0-dense?

It is interesting to compare Question 6 with Question 7; denseness and discreteness can be viewed as opposite properties. Our state of knowledge is so poor (this could well be the author's own ignorance) that we do not know what to expect from a generic subgroup. In fact, the situation is quite interesting even for Lie groups: let G be a connected real Lie group, $k > 1$, and $D^k(G) = \{(g_1, \dots, g_k) \in G^k \mid g_1, \dots, g_k$ generate a discrete subgroup of $G\}$. It is proved in **[Wi1]** that *both $D^k(G)$ and its compliment have positive measure when G is not amenable. If G is amenable, then the situation is also very interesting, namely, there exists n such that $D^k(G)$ is of measure zero for $k > n$ and has complement of measure zero otherwise.*

As emphasized in the previous section, the methods of [**A2**] fail short in effectively understanding C^1-discreteness because although quite involved and subtle, the method essentially is based on the concept of growth. However, just the understanding of C^0-discreteness has produced important results: we have applied Theorem B-C in solving an old problem extending the classical result of Hölder.

Let us recall a classical result (essentially due to Hölder, cf.[**N3**]) that if Γ is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ such that every nontrivial element acts freely then Γ is Abelian. A natural question to ask is what if every nontrivial element has at most N fixed points where N is a fixed natural number. In the case of $N = 1$, we do have a complete answer to this question: it has been proved by Solodov (not published), Barbot [**Ba**], and Kovacevic [**Kov**] that in this case the group is metaabelian, in fact, it is isomorphic to a subgroup of the affine group $\mathrm{Aff}(\mathbb{R})$. (see [**FF1**] for the history of this result, where yet another nice proof is presented).

Applying our results to C^0-discrete subgroups of $\mathrm{Diff}(I)$, we have obtained the following results [**A3**].

Theorem 1.1. *Let Γ be a subgroup of $\mathit{Diff}_+^{1+\epsilon}(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is solvable.*

Assuming a higher regularity on the action we obtain a stronger result.

Theorem 1.2. *Let Γ be a subgroup of $\mathit{Diff}_+^2(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is metaabelian.*

An important tool in obtaining these results is provided by Theorems B-C from [**A2**]. Theorem B (Theorem C) states that a non-solvable (non-metaabelian) subgroup of $\mathrm{Diff}\,_+^{1+\epsilon}(I)$ (of $\mathrm{Diff}\,_+^{2}(I)$) is non-discrete in C^0 metric. Existence of C^0-small elements in a group provides effective tools in tackling the problem. Such tools are absent for less regular actions, and the problem for $\mathrm{Homeo}_+(I)$ (even for $\mathrm{Diff}_+(I)$) still remains open.

2. **Margulis-Zassenhaus Lemma**

One of the major questions that we are interested in is the study of Margulis-Zassenhaus Lemma. This lemma (discovered by H.Zassenhaus in 1938, and later rediscovered by G.Margulis in 1968) states that in a connected Lie group H there exists an open non-empty neighborhood U of the identity such that any discrete subgroup generated by elements from U is nilpotent (see [**Ra**]). For example, if H is a simple Lie group (such as $SL_2(\mathbb{R})$), and $\Gamma \leq H$ is a lattice, then Γ cannot be generated by elements too close to the identity.

In [**A2**], we prove weak versions of Margulis-Zassenhaus Lemma for the group $\mathrm{Diff}_+(I)$. It follows from the results of [**A1**], as remarked there, that the lemma does not hold either for $\mathrm{Diff}_+(I)$ or for $\mathrm{Homeo}_+(I)$, in C^1 and C^0 metrics respectively.

In the increased regularity the lemma still fails: given an arbitrary open neighborhood U of the identity diffeomorphism in $\mathrm{Diff}_+(I)$, it is easy to find two C^∞ "bump functions" in U which generate a discrete group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ which is not nilpotent; thus the lemma fails for $\mathrm{Diff}_+^\infty(I)$.

However, there is a strong evidence (especially in light of the result of É.Ghys mentioned in the previous section) that the answer to the following question might indeed be positive.

Question 8. Does Margulis-Zassenhaus Lemma hold for C^1-discrete subgroups of $\mathrm{Diff}_+^\omega(I)$?

Let us point out the following very interesting property of simple Lie groups (the proof of which uses Margulis-Zassenahus Lemma):

Stability of Generators: We say a connected topological group G has the stability of the generators property (SGP) if the following holds: if $g_1, \dots, g_n$ generate a dense subgroup of G then there exists an open non-empty neighborhood U of identity such that for all $h_i \in g_i U, 1 \leq i \leq n$, the subgroup generated by $h_1, \dots, h_n$ is also dense in G.

Using Margulis-Zassenhaus Lemma, it is indeed not difficult to prove this property for simple Lie groups.

Question 9. Does $\mathrm{Diff}_+(I)$ have SGP in C^1-metric?

We can answer this question for $\mathrm{Homeo}_+(I)$ in C^0-metric. The answer turns out to be negative. In fact, we prove [**A4**] the following concrete statement which is interesting in itself (in the statement of the theorem, $||.||_0$ denotes the C^0 norm).

Proposition 2.1. *Let $\Gamma \leq \mathrm{Homeo}_+(I)$ be a finitely generated subgroup generated by homeomorphisms $g_1, \dots, g_s$. Then for all $\epsilon > 0$, there exist homeomorphisms $h_1, \dots, h_s$ such that $||h_i - g_i||_0 < \epsilon, 1 \leq i \leq s$, and* *the group generated by $h_1, \dots, h_s$ is C^0-strongly discrete, moreover, it is isomorphic to Γ.*

Let us mention the definition of C^0-strong discreteness from [**A1**]: *a subgroup Γ is C^0-strongly discrete if there exists $\delta > 0$ and $x_0 \in (0,1)$ such that $|g(x_0) - x_0| > \delta$ for all $g \in \Gamma \backslash \{1\}$.*

Similarly, one can define the notion of C^1-strong discreteness [**A1**]: *a subgroup Γ is C^1-strongly discrete if there exists $\delta > 0$ and $x_0 \in (0,1)$ such that $|g'(x_0) - 1| > \delta$ for all $g \in \Gamma \backslash \{1\}$.*

Notice that a C^0-strongly (C^1-strongly) discrete subgroup is C^0-discrete (C^1-discrete). In light of Proposition 2.1, it is natural to ask the same question for the group $\mathrm{Diff}_+(I)$.

Question 10. Let $\Gamma \leq \mathrm{Diff}_+(I)$ be a finitely generated subgroup generated by diffeomorphisms $g_1, \dots, g_s$. Then is there a C^1-strongly discrete subgroup (or just C^1-discrete subgroup) generated by diffeomorphisms from an arbitrarily small C^1-neighborhoods of $g_1, \dots, g_s$?

Proposition 2.1 implies that any finitely generated subgroup of $\mathrm{Homeo}_+(I)$ admits a C^0-strongly discrete faithful representation in $\mathrm{Homeo}_+(I)$. The following related question is borrowed from [**A1**]:

Question 11. Is there a finitely generated group Γ which admits a faithful representation in $\mathrm{Diff}_+(I)$ but does not admit a C^1-discrete faithful representation?

It is worth mentioning that, in a connected Lie group, the answer is always positive. For example, the group $\mathbb{Z} \wr \mathbb{Z}$ embeds in $GL(2, \mathbb{R})$ while it does not embed discretely in any connected Lie group.

Question 12. In regard to the Margulis-Zassenhaus Lemma, it is interesting to ask a reverse question, i.e. given an arbitrary open neighborhood U of the identity in $\mathrm{Diff}_+(I)$, is it true that any finitely generated torsion free nilpotent group Γ admits a faithful discrete representation in $\mathrm{Diff}_+(I)$ generated by elements from U?

In [**FF2**], it is proved that any such Γ (i.e. any finitely generated torsion-free nilpotent group) does admit a faithful representation into $\mathrm{Diff}_+(I)$ generated by diffeomorphisms from U. Also, it is proved in [**N4**] that any finitely generated torsion-free nilpotent subgroup of $\mathrm{Diff}_+(I)$ indeed can be conjugated to a subgroup generated by elements from U.

3. **Dense subgroups of Diff(I)**

Discreteness and denseness can be viewed as opposite properties. Dense subgroups capture the (algebraic) content of the ambient group quite strongly while discrete subgroups may remain degenerate in the background of the ambient group unless we demand some uniformity. In the context of connected linear Lie groups, under slight technical conditions, a lattice (a discrete subgroup of finite covolume) turns out to be dense but in the (very weak) Zariski topology. Thus lattices (which also strongly capture the algebraic and geometric content of the ambient Lie group) can also be viewed as dense subgroups in a weaker topology.

For Lie groups, (among the myriad of mostly well studied questions) it is often interesting to ask if a given finitely generated group Γ admits a faithful discrete or dense representation. It is also interesting to ask if the same finitely generated group Γ admits a faithful discrete representation and another faithful dense representation. (if $\Gamma = \mathbb{F}_2$, the answer to the latter question turns out be positive for any non-compact simple Lie group).

For the group $\mathrm{Diff}_+(I)$, questions about discrete subgroups often time lead naturally to a question about dense subgroups (e.g. compare Questions 3 and 4, or see Question 6). In fact, since both discreteness and denseness are poorly understood, if a subgroup $\Gamma \leq \mathrm{Diff}_+(I)$ is non-discrete, one is often tempted to ask is Γ perhaps dense?

The following theorem indicates how far is the group $\mathrm{Diff}_+(I)$ from being solvable. (thus, $\mathrm{Diff}_+(I)$ cannot be possibly viewed as a relative of solvable Lie groups).

Theorem 3.1. [**A4**] *A finitely generated C^0-dense subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ is not elementary-amenable.*

In the case of solvable groups, the proof of Theorem 3.1 is somewhat easier; we show that any finitely generated dense subgroup has infinite girth thus it cannot be solvable (see [**A5**] for the definition of girth).

One approach to studying denseness is to consider an easier notion of dynamical transitivity. In fact, denseness can also be viewed as a dynamical k-transitivity, when $k = \infty$.

We call a subgroup $\Gamma \leq \mathrm{Homeo}_+(I)$ *dynamically k-transitive* if for all $z_1, \dots, z_k \in (0, 1)$, where $z_1 < z_2 < \dots < z_k$ and for all open non-empty intervals $I_1, I_2, \ldots, I_k$ where $x < y$ if $x \in I_p, y \in I_q, 1 \leq p < q \leq k$, there exists $\gamma \in \Gamma$ such that $\gamma(z_i) \in I_i, 1 \leq i \leq k$.

Dynamical transitivity (below, for short, we will call it transitivity) turns out to be quite an interesting notion. Besides serving as an intermediate step towards density, it is naturally interesting from dynamical point of view - transitivity of a group action may reveal information about the underlying algebraic structures of the group. But also, transitivity may play a role of a substitute for ergodicity (if one is aiming to study discrete group actions on "homogenous spaces" of $\mathrm{Diff}_+(I)$; expectedly, even in the absence of Haar measure in the infinite-dimensional group $\mathrm{Diff}_+(I)$, one can still hope to develop a rich ergodic theory; we skip relevant discussions here).

Dynamical transitivity is a key ingredient of the proofs of Theorem 1.1-1.2. Proposition 1.5. of [**A3**] proves 1-transitivity of irreducible non-solvable subgroups of $\mathrm{Diff}_+^{1+\epsilon}(I)$ under the condition that every non-trivial element has at most N fixed points. The same method proves 1-transitivity for irreducible non-metaabelian subgroups of $\mathrm{Diff}_+^{\omega}(I)$ under no condition on the number of fixed points.

What about k-transitivity, for $k > 1$? Obtaining higher transitivity results may result in much bigger rewards. We are still in the process of analyzing very elementary questions in this area. The following questions seem very interesting (in all these questions, we will assume that Γ is an irreducible subgroup of $\mathrm{Diff}_+(I)$, i.e. it has no global fixed point):

Question 13. Is there a finitely generated $\Gamma \leq \mathrm{Diff}_+(I)$ which is k-transitive but not $(k+1)$-transitive?

For $k = 1, 2$, one can easily find subgroups of $\mathrm{Aff}(\mathbb{R})$ with the required property. It is interesting to study this question for $k \geq 3$. In regard to this question, and in general, it is also interesting to compare dynamical transitivity with the usual *strict transitivity.* (a group G acting on a set X is called strictly k-transitive if for any two k-subsets $A, B \subseteq X$ there exists $\gamma \in G$ such that $\gamma(A) = B$). In [Re], the author proves a striking result that any strictly 4-transitive subgroup $\Gamma \leq \mathrm{Diff}_+^{\infty}(\mathbb{S}^1)$ is indeed dense in $\mathrm{Homeo}_+(\mathbb{S}^1)$. Thus, for a circle, in C^{∞} regularity, the answer to Question 13 (in the strictly transitive version) is indeed negative for $k \geq 4$.

Question 14. Let $k \geq 3$. For what values of k does there exist a non-metaabelian subgroup $\Gamma \leq \mathrm{Diff}_+^{\omega}(I)$ which is k-transitive but not $(k+1)$-transitive?

One can ask a similar question even for more specific subgroups:

Question 15. Let $\Gamma \cong \mathbb{F}_2 \leq \mathrm{Diff}_+^\omega(I)$. Is Γ k-transitive for all k?

Notice that being k-transitive for all k is equivalent to C^0-density.

As noted earlier, transitivity, as a dynamical property of a group action, may reveal algebraic properties of groups. It is especially intriguing to understand how transitive can a solvable or amenable group action be. We do not know the answer even to the following very simple question.

Question 16. Is there a finitely generated solvable $\Gamma \leq \mathrm{Diff}_+(I)$ which is 3-transitive? Is there a 3-transitive subgroup Γ of $\mathrm{Diff}_+(I)$ which is not C^0-dense?

In connection with Question 16, let us recall that finitely generated solvable groups cannot be C^0-dense.

It is also interesting to study C^1-dense subgroups in $\mathrm{Diff}_+(I)$; it is natural to expect that the consequences of C^1-density is much stronger, compared with C^0-density. However, we do not know the answer to the following question.

Question 17. Is there a finitely generated group which admits a C^1-dense embedding in $\mathrm{Diff}_+(I)$?

Since $\mathrm{Diff}_+(I)$ is homeomorphic to a Banach space $C_0[0,1]$, let us mention that finitely generated groups do admit a cocompact action by isometries in infinite-dimensional Banach spaces, and even in infinite dimensional separable Hilbert space $\mathcal{H}$; in fact, there exists an example of a finitely generated metaabelian group acting by isometries of $\mathcal{H}$ and having a dense orbit (see **[CTV]**).

4. **Further extensions of Hölder's Theorem**

Theorems 1.1 and 1.2 pave way to a possibly further understanding of group actions on I by homeomorphisms in terms of the sizes of the fixed point sets.

If $\Gamma \leq \mathrm{Homeo}_+(I)$ where all non-trivial elements have at most N fixed points then, for $N \leq 1$, one can indeed classify all such groups. For example, as noted earlier, we have the implications

$$N = 0 \Rightarrow \Gamma \text{ is Abelian}$$

and

$$N = 1 \Rightarrow \Gamma \text{ is isomorphic to a subgroup of } \mathrm{Aff}(\mathbb{R})$$

Question 18. Can one classify all subgroups of $\mathrm{Homeo}_+(I)$ (or of $\mathrm{Diff}_+(I)$) such that every non-identity element has at most N fixed points?

If the number of fixed points of non-identity elements of Γ is not uniformly bounded then it is interesting to know how fast this number grows. In fact, given any compact closed manifold M with a diffeomorphism T of M with finitely many fixed points, it is natural to ask how fast the number of fixed points of T^n grows as $n \to \infty$. If $a_n = |Fix(T^n)|, n = 1, 2, \ldots$, the study of the *the Artin-Mazur zeta function* $\zeta_f(z) = exp(\sum_{n=1}^{\infty} \frac{a_n}{n} z^n)$ has drawn considerable attention since 1960s (**[AM], [Ka]**).

More generally, given a finitely generated group Γ of homeomorphisms of M, it is natural to ask how fast the size of the fixed set $Fix(\gamma)$ grows when $n \to \infty$ where $\gamma \in \Gamma$ is an element of word length at most n w.r.t. the fixed generating set.

For the next three questions, for simplicity, we restrict ourselves mostly to the analytic diffeomorphisms of I but these questions make good sense also for any subgroup Γ of $\mathrm{Diff}_+(I)$ where every non-identity element has finitely many fixed points.

Let $\Gamma \leq \mathrm{Diff}_+^{\omega}(I)$ be a finitely generated subgroup with a fixed finite generating set. Let also

$$\omega_n(\Gamma) = \max_{\phi \in B_n(1)} |Fix(\phi)|, n = 1, 2, \ldots$$

where $B_n(1)$ denotes the ball of radius n centered at the identity element $1 \in \Gamma$.

Question 19. Can ω_n grow superexponentially? Can the growth of ω_n be sub-exponential but superpolynomial?

Since the actions of non-solvable subgroups of $\mathrm{Diff}_+^{1+\epsilon}(I)$ cannot admit a universal upper bound on the number of fixed points, we are wondering if this upper bound (i.e. the number ω_n) may go to infinity very slowly. In particular,

Question 20. Can one classify the groups Γ for which ω_n grows linearly? Is there Γ with a sub-linear growth of ω_n? Can one possibly classify the groups Γ for which ω_n grows polynomially?

Question 21. Compute the growth of ω_n for the group $\Gamma_{f,g}$.

Since the universal bound on the number of fixed points implies solvability, it is interesting **if a slow growth can be an indication of amenability**. In more general words, can the growth of ω_n detect amenability? Related to this, it is interesting to study the growth of ω_n for solvable subgroups of $\mathrm{Diff}_+(I)$. (For $\mathrm{Diff}_+^{\omega}(I)$, all solvable subgroups have been classified [**BW**]).

5. Lattices of Diff $_{+}^{1+\epsilon}(I)$

The study of discrete and dense subgroups naturally leads to the notion of a lattice. This notion makes better sense for the group $\mathrm{Diff}_+^{1+\epsilon}(I)$, although formally it can be defined for the group $\mathrm{Diff}_+(I)$ as well.

Definition. A subgroup $\Gamma \leq$*Diff* $_{+}^{1+\epsilon}(I)$ *is called a lattice if it is* C^1*-discrete and* C^0*-dense.*

A reader is invited to think of the similarities with lattices of linear connected Lie groups; their discreteness in the natural underlying Hausdorff topology and their density in the Zariski topology. Recall that, for a connected linear Lie group G, a lattice of G is Zariski dense in it when a) G is semisimple without compact factors *(Borel Density Theorem)*, or b) G is simply connected and nilpotent [**R**]. (The Zariski density result holds for a general connected linear Lie group but under slightly technical conditions.) Of course, a lattice is much more capturing than just being Zariski dense, but C^0 topology is also much richer than the Zariski topology. In [**A4**], we have studied some basic properties of lattices. We hope that, like in Lie groups, lattices of the group $\mathrm{Diff}_+^{1+\epsilon}(I)$ are very tightly related to it both algebraically and geometrically.

For example, in relation with the amenability problem, we expect that all lattices of $\mathrm{Diff}_+^{1+\epsilon}(I)$ are indeed non-amenable because the ambient group $\mathrm{Diff}_+^{1+\epsilon}(I)$ itself is non-amenable (we would like to caution that in the case of non-locally compact groups, the notion of amenability itself becomes somewhat more delicate). Notice that, by Theorem 3.1, an elementary amenable group cannot be a lattice of $\mathrm{Diff}_+^{1+\epsilon}(I)$. Notice also that, by [**GS**], R.Thompson's group F has a faithful representation in $\mathrm{Diff}_+^{\infty}(I)$ which is simultaneously C^1-discrete and C^0-dense. Hence non-amenability of F should just follow from the fact that it is a lattice of a non-amenable group. At the moment, we do not have clear ideas to prove such a general and very strong conjecture but some basic facts understood on lattices of $\mathrm{Diff}_+^{1+\epsilon}(I)$ seem quite interesting and encouraging. For example, it is indeed true that lattices of $\mathrm{Diff}_+^{1+\epsilon}(I)$ cannot be elementary amenable. [**A4**]

Question 22. Is every lattice of $\mathrm{Diff}_+^{1+\epsilon}(I)$ non-amenable?

The subgroups $\mathrm{Diff}_+^{\omega}(I)$ behave more rigidly and tamely. It is interesting therefore if a lattice of $\mathrm{Diff}_+^{1+\epsilon}(I)$ can sit totally inside a subgroup $\mathrm{Diff}_+^{\omega}(I)$.

Question 23. Is there a subgroup of $\mathrm{Diff}_+^{\omega}(I)$ which is C^1-discrete and C^0-dense?

The group $\mathrm{Diff}_+(I)$, unlike $\mathrm{Diff}_+^{1+\epsilon}(I)$, is a huge source of amenable subgroups. But in $\mathrm{Diff}_+^{\omega}(I)$, we may witness that amenability implies a severe degeneracy.

Question 24. Is it true that any subgroup of $\mathrm{Diff}_+^{\omega}(I)$ is either metaabelian or non-amenable?

In regard to Question 24, let us emphasize that Tits Alternative for $\mathrm{Diff}_+^{\omega}(I)$ (asked by É.Ghys) is still not known.

Many properties of groups can be passed from a Lie group to its lattice, and vice versa. One can ask Question 22 for properties other than amenability. Curiously enough, by *Thurston's Stability Theorem*, $\mathrm{Diff}_+(I)$ does not contain a subgroup with Kazhdan's property (T). (a topological group is amenable and has property (T) at the same time iff it is compact; so the two classes are essentially disjoint). Thus like the ambient group $\mathrm{Diff}_+(I)$, no lattice of it will have property (T). Let us also recall a well known open question: does $\mathrm{Homeo}_+(I)$ have a subgroup with property (T)?

6. **Divisible subgroups of Diff(I)**

The group $\mathrm{Diff}_+(I)$ is indeed a rich source of groups with interesting combinatorial properties. In questions about discreteness, one often encounters group elements which are infinitely divisible (i.e. an element has n-th root for any $n \geq 2$, or for infinitely many $n \geq 2$). It becomes interesting to understand if the n-th root converges to the identity (in C^1 metric or even in C^0-metric).

Let us recall that a group G is called *divisible* if any element $g \in G$ has an n-th root for any natural n, i.e. there exists f such that $f^n = g$. An element $h \in G$ is called *infinitely divisible* if for infinitely many $n \geq 2$ there exists $h_n \in G$ such that $h_n^n = h$.

$\mathbb{Q}$ is a natural example of a divisible group although it is not easy to think a finitely generated non-Abelian divisible group (to our knowledge, the first such example has been constructed by V.Guba [**Gu**]).

Despite exhibiting subgroups with interesting and even wild combinatorial properties, $\mathrm{Diff}_+(I)$ contains no finitely generated divisible subgroup. This immediately follows from *Thurston's Stability Theorem* which states that *any finitely generated subgroup of* $\mathrm{Diff}_+(I)$ *is indicable, i.e. it has a quotient isomorphic to* $\mathbb{Z}$. In [**Be**], it is shown that the group $G = \langle a, b, c \mid a^2 = b^3 = c^7 = abc\rangle$ embeds in $\mathrm{Homeo}_+(I)$. On the other hand, G is perfect. Thus Thurston's Stability Theorem fails for $\mathrm{Homeo}_+(I)$, and the following question becomes interesting.

Question 25. Does $\mathrm{Homeo}_+(I)$ have a finitely generated divisible subgroup?

One is very tempted to conjecture that the answer to Question 25 should be negative, but it is not clear what tools to use. One might try to expand on the ideas of Thurston's proof but there are significant difficulties.

Question 26. Does $\mathrm{Diff}_+(I)$ have a finitely generated subgroup with an infinitely divisible non-trivial element f and a sequence f_{n_k} of n_k-th roots for an increasing sequence n_k (i.e. $n_1 < n_2 < \dots$ and $f_{n_k}^{n_k} = f, \forall k \in \mathbb{N}$) such that the sequence f_{n_k} does not converge to identity in C^0-metric?

The answer to Question 26 is positive in $\mathrm{Homeo}_+(I)$, and probably negative in $\mathrm{Diff}_+^{\omega}(I)$. For Lie groups, it is not difficult to see that there is no finitely generated divisible subgroup in a Lie group, moreover, the analogue of Question 26 has a negative answer.

A non-trivial homeomorphism has a square root in $\mathrm{Homeo}_+(I)$, in fact a whole continuum of them! (I have learned this from Matthew Brin). But a situation is quite different and in some instances very complicated in the groups $\mathrm{Diff}_+(I)$ and $PL_+(I)$. Both the existence and uniqueness (of square roots, and more generally, of the n-th roots) questions are interesting.

7. **Subgroups of** $PL_+(I)$

In recent years, significant advances have been made in understanding (and in some instances even classifying) subgroups of $\mathrm{PL}_+(I)$ from both algebraic and geometric points of view **[BS], [Br1], [Br2], [Bl1], [Bl2]**. The representations of subgroups of $\mathrm{PL}_+(I)$ in $\mathrm{Diff}_+(I)$ though are still very poorly understood.

By the result of Ghys-Sergiescu **[GS]**, R.Thompson's group F, being perhaps the most interesting and fascinating example of a finitely generated subgroup of $\mathrm{PL}_+(I)$, indeed does admit an embedding in $\mathrm{Diff}_+^{\infty}(I)$. For this, the authors start with the standard PL representation and introduce a certain "change of variables" to smooth out the representation to make it C^{∞}-regular. The standard generators of F cooperate very successfully in carrying out this process, and already for some close relatives of F the argument breaks down. Thus the following question becomes extremely intriguing.

Question 27. Does every finitely generated subgroup of $\mathrm{PL}_+(I)$ admit a faithful representation in $\mathrm{Diff}_+(I)$?

The answer is expected to be negative although we have very little idea at the moment about how can one approach to this problem. For a finitely generated group Γ, having a faithful embedding in $\mathrm{PL}_+(I)$ implies very strong structural rigidity from algebraic point of view. Despite this rigidity, $\mathrm{PL}_+(I)$ seems to be rich enough with finitely generated subgroups that not all of them have embeddings in $\mathrm{Diff}_+(I)$. The regularity of the embedding (if it exists) is also an interesting question. In fact it is useful to consider the following chain of subgroups.

$$\begin{array}{l}\mathrm{Homeo}_+(I) \supset \mathrm{Diff}_+(I) \supset \mathrm{Diff}_+^{1+\epsilon}(I) \supset \mathrm{Diff}_+^2(I) \supset \mathrm{Diff}_+^{\infty}(I) \supset \mathrm{Diff}_+^{\omega}(I) \\ \quad \cup \\ \quad \mathrm{PL}_+(I)\end{array}$$

For each of the inclusion in the above diagram, it is interesting to find finitely generated groups in the ambient group which do not have a faithful representation in the subgroup. Examples of finitely generated groups which embed in $\mathrm{Homeo}_+(I)$ but not in $\mathrm{Diff}_+(I)$ are known **[Be], [N1]**. On the other hand, Theorem 0.1 (combined results of Plante-Thurston and Farb-Franks) provide rich source of examples of finitely generated groups (namely, any f.g. torsion-free non-Abelian nilpotent group) embedding in $\mathrm{Diff}_+(I)$ but not in $\mathrm{Diff}_+^2(I)$. Also, combining Farb-Franks result with the result of A.Navas **[N1]** one obtains examples of f.g. nilpotent groups embedding in $\mathrm{Diff}_+(I)$ but not in

$\mathrm{Diff}_+^{1+\epsilon}(I)$. On the other hand, there are many examples of f.g. groups which embed in $\mathrm{Diff}_+^{\infty}(I)$ but not in $\mathrm{Diff}_+^{\omega}(I)$ (by the result of É.Ghys [**G2**], any solvable subgroup of $\mathrm{Diff}_+^{\omega}(I)$ is metabelian). By the above mentioned work of Ghys-Sergiescu [**GS**], R.Thompson's group F also is a group like this. One can obtain a huge source of solvable groups with the required property using the classification result of Burslem and Wilkinson [**BW**].

For the inclusions $G_1 \supset G_2$ of the above diagram, it is also interesting to find a f.g. subgroup of $\mathrm{PL}_+(I)$ which embeds in the ambient group G_1 but does not embed in the subgroup G_2. The following question is interesting for any finitely generated group, not just for finitely generated subgroups of $\mathrm{PL}_+(I)$.

Question 28. Is there a finitely generated group Γ which embeds in $\mathrm{Diff}_+^2(I)$ but not in $\mathrm{Diff}_+^3(I)$?

It is immediate to see that a non-Abelian subgroup of $\mathrm{PL}_+(I)$ is never C^0-discrete. But C^1-discreteness is not well understood at the moment. To be precise, let us define the following C^1-metric d in $\mathrm{PL}_+(I)$: for any two maps $f, g \in \mathrm{PL}_+(I)$, let

$$d(f,g) = \sup_{x \in (0,1)\backslash(S(f)\cup S(g))} |f'(x) - g'(x)|$$

where $S(h)$ denotes the (finite) set of singularities of a map $h \in \mathrm{PL}_+(I)$.

Notice that, if for all $f \in \Gamma \leq \mathrm{PL}_+(I)$, the slope of f at every point of $(0,1)\backslash S(f)$ belongs to a discrete multiplicative subgroup of $\mathbb{R}_+$ (e.g. R.Thompson's group F in its standard representation in $\mathrm{PL}_+(I)$) then Γ is C^1-discrete. Thus there are plenty of examples of interesting discrete subgroups of $\mathrm{PL}_+(I)$. However, unlike the case of $\mathrm{Diff}_+(I)$ it is apparently not easy to find a non-discrete subgroup.

Question 29. Is there a finitely generated non-discrete subgroup of $\mathrm{PL}_+(I)$ in C^1 metric?

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