On groups of diffeomorphisms of the interval with finitely many fixed points I

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Abstract: We strengthen the results of [1], consequently, we improve the claims of [2] obtaining the best possible results. Namely, we prove that if a subgroup Γ of $\mathrm{Diff}_+(I)$ contains a free semigroup on two generators then Γ is not C_0-discrete. Using this we extend the Hölder's Theorem in $\mathrm{Diff}_+(I)$ classifying all subgroups where every non-identity element has at most N fixed points. By using the concept of semi-archimedean groups, we also show that the classification picture fails in the continuous category.

1. Introduction

Throughout this paper we will write $\mathbf{\Phi}$ (resp. $\mathbf{\Phi}^{\mathrm{diff}}$) to denote the class of subgroups of $\Gamma \leq \mathrm{Homeo}_+(I)$ (resp. $\Gamma \leq \mathrm{Diff}_+(I)$) such that every non-identity element of Γ has finitely many fixed points. Let us point out immediately that any subgroup of $\mathrm{Diff}_+^{\omega}(I)$ - the group of orientation preserving analytic diffeomorphisms of I - belongs to $\mathbf{\Phi}$. In fact, many of the major algebraic and dynamical properties of subgroups of $\mathrm{Diff}_+^{\omega}(I)$ is obtained solely based on this particular property of analytic diffeomorphisms having only finitely many fixed points. Interestingly, groups in $\mathbf{\Phi}$ may still have both algebraic and dynamical properties not shared by any subgroup of $\mathrm{Diff}_+^{\omega}(I)$. In particular, not every group in $\mathbf{\Phi}$ is conjugate to a subgroup of $\mathrm{Diff}_+^{\omega}(I)$.

For a non-negative integer $N \geq 0$, we will also write $\mathbf{\Phi}_N$ (resp. $\mathbf{\Phi}_N^{\mathrm{diff}}$) to denote the class of subgroups of $\Gamma \leq \mathrm{Homeo}_+(I)$ (resp. $\Gamma \leq \mathrm{Diff}_+(I)$) such that every non-identity element of Γ has at most N fixed points in the interval $(0, 1)$.

Characterizing $\mathbf{\Phi}_N$ for an arbitrary N is a major open problem solved only for values $N = 0$ and $N = 1$: Hölder's Theorem states that any subgroup of $\mathbf{\Phi}_0$ is Abelian, while Solodov's Theorem states [1] that any subgroup of $\mathbf{\Phi}_1$ is metaabelian, in fact, it is isomorphic to subgroup of $\mathrm{Aff}_+(\mathbb{R})$ - the group of orientation preserving affine homeomorphisms of $\mathbb{R}$.

[1]Solodov's result is unpublished but three independent proofs have been given by Barbot [3], Kovacevic [6], and Farb-Franks [4]

It has been proved in [1] that, for $N \geq 2$, any subgroup of $\mathbf{\Phi}_N^{\text{diff}}$ of regularity $C^{1+\epsilon}$ is indeed solvable, moreover, in the regularity C^2 we can claim that it is metaabelian. The argument there fails short in complete characterization of subgroups of $\mathbf{\Phi}_N^{\text{diff}}, N \geq 2$ even at these increased regularities.

In [7], Navas gives a different proof of this result for groups of analytic diffeomorphisms, namely, it is shown that any group in $\mathbf{\Phi}_N^{\text{diff}}$ of class C^{ω} is necessarily metaabelian.

In this paper, we provide a complete characterization of the class $\mathbf{\Phi}_N^{\text{diff}}$ for an arbitrary N. Our main result is the following

Theorem 1.1. *Let* $\Gamma \leq \text{Diff}_+(I)$ *be an irreducible subgroup, and* $N \geq 0$ *such that every non-identity element has at most N fixed points. Then* Γ *is isomorphic to a subgroup of* $\text{Aff}_+(\mathbb{R})$.

In other words, any irreducible subgroup of $\mathbf{\Phi}_N^{\text{diff}}$ is isomorphic to an affine group. Indeed, we show that, for $N \geq 2$, any irreducible subgroup of $\mathbf{\Phi}_N^{\text{diff}}$ indeed belongs to $\mathbf{\Phi}_1^{\text{diff}}$! Let us point out that there exist metaabelian examples (communicated to the author by A.Navas; a certain non-standard representation of the Baumslag-Solitar group $BS(1,2)$ in $\text{Homeo}_+(I)$) which shows that the class $\mathbf{\Phi}_N$ is indeed strictly larger than the class $\mathbf{\Phi}_1$, for $N \geq 2$. We will present examples of non-metabelian groups from $\mathbf{\Phi}_N, N \geq 2$ thus constructing examples from $\mathbf{\Phi}_N, N \geq 2$ which are algebraically non-isomorphic to subgroups of $\mathbf{\Phi}_1$.

2. C_0-discrete subgroups of $\text{Diff}_+(I)$: strengthening the results of [1]

The main results of [2] are obtained by using Theorems B-C from [1]. Theorem B (Theorem C) states that a non-solvable (non-metaabelian) subgroup of $\text{Diff}_+^{1+\epsilon}(I)$ (of $\text{Diff}_+^2(I)$) is non-discrete in C_0 metric. Existence of C_0-small elements in a group provides effective tools in tackling the problem. Theorems B-C are obtained by combining Theorem A in [1] by the results of Szekeres, Plante-Thurston and Navas. Theorem A states that for a subgroup $\Gamma \leq \text{Diff}_+(I)$, if $[\Gamma, \Gamma]$ contains a free semigroup in two generators then Γ is not C_0-discrete. In the proof of Theorem A, the hypothesis that the generators of the free semigroup belong to the commutator subgroup $[\Gamma, \Gamma]$ is used only to deduce that the derivatives of both of the generators at either of the end points of the interval I equal 1. Thus we have indeed proved the following claim:

Let $\Gamma \leq \mathrm{Diff}_+(I)$ be a subgroup containing a free semigroup in two generators f, g such that either $f'(0) = g'(0) = 1$ or $f'(1) = g'(1) = 1$. Then Γ is not C_0-discrete, moreover, there exists non-identity elements in $[\Gamma, \Gamma]$ arbitrarily close to the identity in C_0 metric.

In this section, we make a simple observation which strengthens Theorem A further, namely, the condition "$[\Gamma, \Gamma]$ contains a free semigroup" can be replaced altogether with "Γ contains a free semigroup" (i.e. without demanding the extra condition "either $f'(0) = g'(0) = 1$ or $f'(1) = g'(1) = 1$".

Theorem 2.1 (Theorem A$'$). *Let $\Gamma \leq \mathrm{Diff}_+(I)$ be a subgroup containing a free semigroup in two generators. Then Γ is not C_0-discrete, moreover, there exists non-identity elements in $[\Gamma, \Gamma]$ arbitrarily close to the identity in C_0 metric.*

In the proof of Theorems B-C, if we use Theorem A$'$ instead of Theorem A we obtain the following stronger versions.

Theorem 2.2 (Theorem B$'$). *If a subgroup $\Gamma \leq \mathrm{Diff}_+^{1+\epsilon}(I)$ is C_0-discrete then it is virtually nilpotent.*

Theorem 2.3 (Theorem C$'$). *If a subgroup $\Gamma \leq \mathrm{Diff}_+^2(I)$ is C_0-discrete then it is Abelian.*

Theorem A$'$ is obtained from the proof of Theorem A by a very slight modification. Let us first assume that Γ is irreducible, i.e. it has no fixed point on $(0, 1)$. Let $f, g \in \Gamma$ generate a free semigroup on two generators. If $f'(0) = g'(0) = 1$ or $f'(1) = g'(1) = 1$ then the claim is already proved in [1], otherwise, without loss of generality we may assume that $f'(1) < 1$ and $g'(1) < 1$.

Let also $\epsilon, N, \delta, M, \theta$ be as in the proof of Theorem A in [1], except we demand that $1 < \theta_N < \sqrt[8N]{1.9}$ (instead of $1 < \theta_N < \sqrt[2N]{2}$), and instead of the inequality $\frac{1}{\theta_N} < \phi'(x) < \theta_N$, we demand that

$$\max_{x,y \in [1-\delta, 1]} \left(\frac{\phi'(x)}{\phi'(y)}\right)^8 < \theta_N,$$

where $\phi \in \{f, g, f^{-1}, g^{-1}\}$. In addition, we also demand that for all $x \in [1 - \delta, 1]$, we have $f(x) > x$ and $g(x) > x$.

Then we let $W = W(f, g), \alpha, \beta \in \Gamma$ be as in the proof of Theorem A. We may also assume that (by replacing (α, β) with $(\alpha\beta, \beta\alpha)$ if necessary), $\alpha'(0) = \beta'(0) = \lambda < 1$.

Now, for every $n \in \mathbb{N}$, instead of the set

$$\mathbb{S}_n = \{U(\alpha,\beta)\beta\alpha \mid U(\alpha,\beta) \text{ is a positive word in } \alpha,\beta \text{ of length at most } n\}$$

we consider the set

$$\mathbb{S}'_n = \{U(\alpha,\beta)\beta\alpha \in \mathbb{S}_n \mid \text{sum of exponents of } \alpha \text{ in } U(\alpha,\beta) \text{ equals } [\frac{n}{2}]\}$$

Previously, we had the crucial inequality $|\mathbb{S}_n| \geq 2^n$ for all n but now we have the inequality $|\mathbb{S}'_n| \geq (1.9)^n$ for sufficiently big n. Let us also observe that, for any interval J in $(1-\delta, 1)$, and for all $g \in \mathbb{S}'_n$, we will have the inequality $|g(J)| < \lambda^n (\theta_N)^{\frac{1}{8}n}$. Then for some sufficiently big n the following conditions hold:

(i) there exist $g_1, g_2 \in \mathbb{S}_n$ such that $g_1 \neq g_2$, and

$$|g_1 W(x_i) - g_2 W(x_i)| < \frac{1}{\sqrt[2N]{1.9}^n}, 1 \leq i \leq N-1,$$

(ii) $M^{2m+4}(\theta_N)^{4n} \frac{1}{\sqrt[2N]{1.9}^n} < \epsilon$,

where $x_i = \frac{i}{N}, 0 \leq i \leq N$.

The rest of the proof goes exactly the same way by replacing $\mathbb{S}_n$ with $\mathbb{S}'_n$: letting again $h_1 = g_1 W, h_2 = g_2 W$, we obtain that $|h_1^{-1}h_2(x) - x| < 2\epsilon$ for all $x \in [0,1]$. Since ϵ is arbitrary, we obtain that Γ is not C_0-discrete. On the other hand, by definition of $\mathbb{S}'_n$ we have $h_1^{-1}h_2 \in [\Gamma,\Gamma]$. If Γ is not irreducible then it suffices to observe that there exists only finitely many intervals $I_1, \ldots, I_m$ in $(0,1)$ such that Γ fixes the endpoints of I_j but no other point inside I_j, moreover, $\sum_{1 \leq j \leq m} |I_j| > 1 - 2\epsilon$

□

3. Extension of Hölder's Theorem in $\mathrm{Diff}_+(I)$

Let us point out that the following theorem follows from the proof of Theorem 0.1 and Theorem 0.2 in [2].

Theorem 3.1. *Let $N \geq 0$ and Γ be an irreducible group in $\mathbf{\Phi}_N^{\mathrm{diff}}$ such that $[\Gamma,\Gamma]$ contains diffeomorphisms arbitrarily close to the identity in C_0 metric. Then Γ belongs to $\mathbf{\Phi}_1^{\mathrm{diff}}$ thus it is isomorphic to a subgroup of the affine group $\mathrm{Aff}_+(\mathbb{R})$.*

The method of [2] does not allow to obtain a complete classification of subgroups of $\mathbf{\Phi}_N^{\mathrm{diff}}$ primarily because existence of non-discrete subgroups in $\mathrm{Diff}_+^{1+\epsilon}(I)$ (in $\mathrm{Diff}_+^2(I)$) is guaranteed only for non-solvable (non-metaabelian) groups. Within the class of solvable (metaabelian) groups the method is inapplicable.

Now, by Theorem A$'$, we can guarantee the existence of non-discreteness in the presence of a free semigroup. On the other hand, the property of containing a free semigroup on two generators is generic only in $C^{1+\epsilon}$ regularity; more precisely, any non-virtually nilpotent subgroup of $\mathrm{Diff}_+^{1+\epsilon}(I)$ contains a free semigroup on two generators. Just in C^1-regularity, $\mathrm{Diff}_+(I)$ has many non-virtually nilpotent subgroups (e.g. subgroups of intermediate growth) without free semigroups. (see [8])

The next proposition indicates a strong distinctive feature for groups of $\mathbf{\Phi}$, and supplies free semigroups for all non-Abelian subgroups in $\mathbf{\Phi}_N, N \geq 1$.

Proposition 3.2. *Any subgroup in class $\mathbf{\Phi}$ is either Abelian or contains a free semigroup on two generators.*

Corollary 3.3. For any $N \geq 0$, a subgroup of $\mathbf{\Phi}_N$ is either Abelian or contains a free semigroup.

Remark 3.4. Let us point out that any group Γ in $\mathbf{\Phi}$ is bi-orderable. A bi-order can be given as follows: for $f, g \in \Gamma$, we let $f < g$ iff $f(x) < g(x)$ in some interval $(0, \delta)$. Proposition 3.2 shows that the converse is far from being true, i.e. not every finitely generated bi-orderable group embeds in $\mathbf{\Phi}$. For example, it is well known that every finitely generated torsion-free nilpotent group is bi-orderable hence it embeds in $\mathrm{Homeo}_+(I)$ (by the result of [5] it embeds into $\mathrm{Diff}_+(I)$ as well); on the other hand, a finitely generated nilpotent group does not contain a free semigroup on two generators.

We need the following well known notion.

Definition 3.5. Let $f, g \in \mathrm{Homeo}_+(I)$. We say the pair (f, g) is *crossed* if there exists a non-empty open interval $(a, b) \subset (0, 1)$ such that one of the homeomorphisms fixes a and b but no other point in (a, b) while the other homeomorphism maps either a or b into (a, b).

It is a well known folklore result that if (f, g) is a crossed pair then the subgroup generated by f and g contains a free semigroup on two generators (see [9]).

Proof of Proposition 3.2. We may assume that Γ is irreducible. If Γ acts freely then by Hölder's Theorem it is Abelian and we are done. Otherwise, there exists a point $p \in (0, 1)$ which is fixed by some non-identity element f of Γ. Since Γ is not irreducible, there exists g which does not fix p. Let p_+ be the biggest fixed point of g less than p, and p_- be the smallest fixed point of g bigger than p. If at least one of the points p_+, p_- is not fixed by f then either the pair (f, g) or (f^{-1}, g) is crossed.

Now assume that both p_+, p_- are fixed by f. Without loss of generality we may also assume that $g(x) > x$ for all $x \in (p_+, p_-)$. Let q_- be the smallest fixed point of f bigger than p_-, and q_+ be the biggest fixed point of f smaller than p_+. (we have $q_- \leq q_+$ but it is possible that q_- equals q_+). Then there exists $n \geq 1$ such that $g^n(q_-) > q_+$. Then either the pair $(g^n f g^{-n}, f)$ or the pair $(g^n f^{-1} g^{-n}, f)$ is crossed (in the interval $(a, b) = (q_+, p_+)$). $\square$

4. Semi-archimedian groups

It is a well known fact that any subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is left-orderable. Conversely, one can realize any countable left-orderable group as a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ (see [9]). Despite such an almost complete and extremely useful characterization of left-orderable groups, when presented algebraically (or otherwise) it can be difficult to decide if the group does admit a left order at all, and if yes, then are there many left orders?

For example, it is true that a semi-direct product of a left-orderable group with another left-orderable group is still let-orderable. In fact, if the groups G, H admit left orders $\prec_1, \prec_2$ respectively, then one can put a left order $\prec$ on $G \ltimes H$ by letting, $(g_1, h_1) \prec (g_2, h_2)$ iff either $g_1 \prec_1 g_2$ or $g_1 = g_2, h_1 \prec_2 h_2$. This left order is quite straightforward; here, *G is dominant over H* and because G is the acting group, one checks directly that the linear order $\prec$ is indeed left-invariant. It is sometimes more interesting (and needed for our purposes in this paper) to make H dominant over G; one can do this if the action of G on H preserves the left order of H. We materialize this in the following

Lemma 4.1. *Let a group G_1 acts on a group G_2 by automorphisms. Let $\prec_1, \prec_2$ be left orders on G_1, G_2 respectively, and assume that the action of G_1 on G_2 preserves the left order [i.e. if $g \in G_1, x_1, x_2 \in G_2, x_1 \prec_2 x_2$ then $g(x_1) \prec_2 g(x_2)$].*

Then there exists a left order $<$ in $G_1 \ltimes G_2$ which satisfies the following conditions:

1) *if $g_1, f_1 \in G_1, g_1 \prec_1 f_1$ then $(g_1, 1) < (f_1, 1)$;*

2) *if $g_2, f_2 \in G_2, g_2 \prec_2 f_2$ then $(1, g_2) < (1, f_2)$;*

3) *if $g_1 \in G_1 \backslash \{1\}, g_2 \in G_2 \backslash \{1\}, 1 \prec_2 g_2$, then $(g_1, 1) < (1, g_2)$.*

Proof. We define the left order on $G_1 \ltimes G_2$ as follows: given $(g_1, f_1), (g_2, f_2) \in G_1 \ltimes G_2$ we define $(g_1, f_1) < (g_2, f_2)$ iff either $f_1 \prec_2 f_2$ or $f_1 = f_2, g_1 \prec_1 g_2$. Then the claim is a direct check. $\square$

The left order $<$ on the semidirect product $G_1 \ltimes G_2$ constructed in the proof of the lemma will be called the *extension of $\prec_1$ and $\prec_2$*.

Let G be a group with a left order $<$. G is called *Archimedean* if for any two positive elements $f, g \in G$, there exists a natural number n such that $g^n > f$. In other words, for any positive element f, the sequence $(f^n)_{n \geq 1}$ is *strictly increasing* and *unbounded*.[2] It is a classical result, proved by Hölder, that Archimedean group are necessarily Abelian, moreover, they are always isomorphic to a subgroup of $\mathbb{R}$. In fact, the notion of Archimedean group arises very naturally in proving the fact that any freely acting subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is Abelian, first, by showing that such a group must be Archimedean, and then, by a purely algebraic argument (due to Hölder), proving that *Archimedean $\Rightarrow$ Abelian.*

It turns out one can generalize the notion of Archimedean groups to obtain algebraic results of similar flavor for subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ which do not necessarily act freely but every non-trivial element has at most N fixed points. Let us first consider the following property.

Definition 4.2. Let G be a group with a left order $<$. We say G satisfies property (P_1) if there exists a natural number M and elements $g, \delta \in G$ such that if the sequence $(g^n)_{n \geq 1}$ is increasing but bounded, and $\delta g^k > g^m$ for all $k, m > M$, then for all $k \geq M$ either the sequence $(g^n \delta g^k)_{n \geq 1}$ or the sequence $(g^{-n} \delta g^k)_{n \geq 1}$ is increasing and unbounded.

[2] In a left-orderable group G, we say a sequence $(g_n)_{n \geq 1}$ is bounded if there exists an element g such that $g^{-1} < g_n < g$ for all $n \geq 1$.

Every Archimedean group clearly satisfies property (P_1) but there are non-archimedean groups too with property (P_1). In fact, it is easy to verify that the metaabelian affine group $\mathrm{Aff}_+(\mathbb{R})$ with the following very natural order does satisfy property (P_1) while not being Archimedean: for any two maps $f, g \in \mathrm{Aff}_+(\mathbb{R})$ we say $f < g$ iff either $f(0) < g(0)$ or $f(0) = g(0), f(1) < g(1)$.

An Archimedean group can be viewed as groups where powers of positive elements *reach infinity*. In groups with property (P_1), the power of a positive element reaches infinity perhaps after an extra *arbitrarily small one time push*, namely if $g \in G$ is positive and $(g^n)_{n\geq 1}$ is still bounded, then for every δ where $\delta g^m > g^k$ for all sufficiently big m, k, either the sequence ${g^n \delta g^m}_{n\geq 1}$ or the sequence ${g^{-n}\delta g^m}_{n\geq 1}$ reaches the infinity. Thus groups with property (P_1) can be viewed as generalization of Archimedean groups. We would like to introduce even a more general property (P_N) for any $N \geq 1$. (Archimedean groups can be viewed as exactly the groups with property (P_0)).

Definition 4.3. Let G be a group with a left order $<$, and N be a natural number. We say G satisfies property (P_N) if there exists a natural number M, the elements $g, \delta_1, \dots, \delta_{N-1} \in G$, and the numbers $\epsilon_1, \dots, \epsilon_{N-1} \in \{-1, 1\}$ such that if, for all $i \in \{1, \dots, N-1\}$ and for all $k_1, \dots, k_{i-1}, k_i \geq M, \epsilon_1, \dots, \epsilon_i \in \{-1, 1\}$,

(i) the sequence $(g^{\epsilon_i n}\delta_{i-1}g^{\epsilon_{i-1}k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n\geq 1}$ is bounded from above, and

(ii) $\delta_i g^{\epsilon_i k_i}\delta_{i-1}g^{\epsilon_{i-1}k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1} > g^{\epsilon_i k_i}\delta_{i-1}g^{\epsilon_{i-1}k_{i-1}} \dots \delta_1 g^{\epsilon_1 k_1}$

then, for some $\epsilon_N \in \{-1, 1\}$, the sequence $(g^{\epsilon_N n}\delta_{N-1}g^{\epsilon_{N-1}k_{N-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n\geq 1}$ is unbounded from above.

Remark 4.4. Similarly, in groups with property (P_N) the power of a positive element may not necessarily reach the infinity but does so after some N arbitrarily small pushes (by $\delta_1, \dots, \delta_N$). Namely, one considers the sequences $g^n, g^{\pm n}\delta_1 g^n, g^{\pm n}\delta_2 g^{\pm n}\delta_1 g^n, \dots, g^{\pm n}\delta_N \dots g^{\pm n}\delta_1 g^n$ and one of them reaches infinity as $n \to \infty$.

Remark 4.5. In the case of $N = 0$, the existence of elements $g_1, \delta_1, \dots, g_{N-1}, \delta_{N-1}$ is a void condition, and one can state condition (P_0) as the existence of an element g_0 such that g_0^n is unbounded; thus groups with property (P_0) are exactly the Archimedean groups.

Definition 4.6. A left ordered group G is called *semi-Archimedean* if it satisfies property (P_N) for some $N \geq 0$.

We will need the following result about semi-Archimedean groups:

Proposition 4.7. *Let G be a countable semi-Archimedean group. Then G has a realization as a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *such that every non-identity element has at most N fixed points.*

Proof. For simplicity, we will first prove the proposition for $N = 1$. (In fact, for the application in the next section, Proposition 4.7 is needed only in the case $N = 1$).

If there exists a smallest positive element in Γ then, necessarily, Γ is cyclic and the claim is obvious. Let $g_1, g_2, \ldots$ be all elements of Γ where $g_1 = 1$. We can embed Γ in $\mathrm{Homeo}_+(\mathbb{R})$ such that the sequence $\{g_n(0)\}_{n\geq 1}$ satisfies the following condition: $g_1(0) = 0$, and for all $n \geq 1$,

(i) if $g_{n+1} > g_i$ for all $1 \leq i \leq n$, then $g_{n+1}(0) = \max\{g_i(0) \mid 1 \leq i \leq n\} + 1$,

(ii) if $g_{n+1} < g_i$ for all $1 \leq i \leq n$, then $g_{n+1}(0) = \min\{g_i(0) \mid 1 \leq i \leq n\} - 1$,

(iii) if $g_i < g_{n+1} < g_j$ and none of the elements $g_1, \ldots, g_n$ is strictly in between g_i and g_j then $g_{n+1}(0) = \frac{g_i(0)+g_j(0)}{2}$.

Then, since there is no smallest positive element in Γ, we obtain that the orbit $O = \{g_n(0)\}_{n\geq 1}$ is dense in $\mathbb{R}$. This also implies that the group Γ for any point $p \in O$ and for any open non-empty interval I, there exists $\gamma \in \Gamma$ such that $\gamma(p) \in I$.

Now assume that some element g of Γ has at least two fixed points. Then for some p, q we have $Fix(g) \cap [p, q] = \{p, q\}$. Without loss of generality, we may also assume that $p > 0$ and $g(x) > x$ for all $x \in (p, q)$. By density of the orbit $\{g_n(0)\}_{n\geq 1}$, there exists $f \in \Gamma$ such that $f(0) \in (p, q)$. Then, for sufficiently big n, we have $\delta = g^{-n}f$ has a fixed point $r \in (p, q)$, moreover, $\delta(x) > x$ for all $x \in (p, r)$.

Then $g^{\epsilon n}$ does not reach infinity for any $\epsilon \in \{-1, 1\}$, in fact, $g^{\epsilon n}(0) < p$ for all $n \geq 1, \epsilon \in \{-1, 1\}$. Then $\{g^{\epsilon_1 n}\delta g^{\epsilon k}\}_{n\geq 1}$ does not reach infinity for any $k \in \mathbb{Z}, \epsilon, \epsilon_1 \in \{-1, 1\}$. Contradiction.

To treat the case of general $N \geq 1$, let us assume that some element $g \in \Gamma$ has at least $N + 1$ fixed points. Then there exists open intervals $I_1 = (a_1, b_1), \ldots, I_{N+1} = (a_{N+1}, b_{N+1}$ such that $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_N < b_N \leq a_{N+1} < b_{N+1}$ and $\{a_1, b_1, \ldots, a_{N+1}, b_{N+1} \subset Fix(g)$. By density of the orbit O, there exist elements $\delta_1, \ldots, \delta_N$ such that $\delta_i(b_i) \in I_{i+1}, 1 \leq i \leq N$. Then for the appropriate choices of $\epsilon_1, \ldots, \epsilon_{N-1} \in \{-1, 1\}$ and for sufficiently big $k_1, \ldots, k_{N_1}$, conditions (i) and (ii) of Definition 4.3 hold, while for any $\epsilon_N \in \{-1, 1\}$, the sequence $(g^{\epsilon_N n}\delta_{N-1}g^{\epsilon_{N-1}k_{N-1}} \dots \delta_1 g^{\epsilon_1 k_1})_{n\geq 1}$ is bounded from above because it lies in I_{N+1}. $\square$

Remark 4.8. Let us emphasize that in this section we did not make an assumption that the groups belong to the class $\mathbf{\Phi}$.

5. A non-affine subgroup of $\mathbf{\Phi}_N$

In this section we will present an irreducible non-affine subgroup Γ from $\mathbf{\Phi}_N$ for $N \geq 2$ thus showing that the classification result of Theorem 1.1 fails in the continuous category. The method for the construction suggests that one can obtain a solvable group of arbitrarily high derived length in $\mathbf{\Phi}_N$ but we would like to emphasize that we do not know any non-solvable example.

The subgroup Γ will be given by a presentation

$$\langle t, s, b \mid tbt^{-1} = b^2, sbs^{-1} = b^2, [t, s] = 1\rangle$$

so it has a relatively simple algebraic structure; it is indeed isomorphic to the semidirect product of $\mathbb{Z}^2$ with the additive group of the ring $D = \mathbb{Z}[\frac{1}{2}]$ where b can be identified with identity of the ring $\mathbb{Z}[\frac{1}{2}]$, s, t can be identified with the standard generators of $\mathbb{Z}^2$, and the action of both t and s on D is by multiplication by 2. However, we will put a left order in it which is not the most natural left order that one considers.

Let $\prec_1$ be the natural left order on $\mathbb{Z}$, and $\prec_2$ be the left order on $\mathbb{Z}[\frac{1}{2}]$ induced by the usual order on $\mathbb{R}$. Notice that the action of $\mathbb{Z}$ on the group $\mathbb{Z}[\frac{1}{2}]$ preserves the left order $\prec_2$. Then we let $<$ be the extension of the left orders $\prec_1$ and $\prec_2$. By Lemma 4.1, $<$ is a left order on Γ. One can check easily that the group Γ satisfies property (P_1).

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