EXTENSION OF HÖLDER'S THEOREM IN Diff $_{+}^{1+\epsilon}(I)$

Azer Akhmedov

Abstract: We prove that if Γ is subgroup of Diff $_{+}^{1+\epsilon}(I)$ and N is a natural number such that every non-identity element of Γ has at most N fixed points then Γ is solvable. If in addition Γ is a subgroup of Diff $_{+}^{2}(I)$ then we can claim that Γ is metaabelian.

It is a classical result (essentially due to Hölder, cf.[N]) that if Γ is a subgroup of $\mathrm{Homeo}_{+}(\mathbb{R})$ such that every nontrivial element acts freely then Γ is Abelian. A natural question to ask is what if every nontrivial element has at most N fixed points where N is a fixed natural number. In the case of $N = 1$, we do have a complete answer to this question: it has been proved by Solodov (not published), Barbot [B], and Kovacevic [K] that in this case the group is metaabelian, in fact, it is isomorphic to a subgroup of the affine group Aff($\mathbb{R}$). (see [FF] for the history of this result, where yet another nice proof is presented).

In this paper, we answer this question for an arbitrary N assuming some regularity on the action of the group.

Our main result is the following theorem.

Theorem 0.1.(Main Theorem) Let $\epsilon \in (0, 1)$ and Γ be a subgroup of Diff $_{+}^{1+\epsilon}(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is solvable.

Assuming a higher regularity on the action we obtain a stronger result

Theorem 0.2. Let Γ be a subgroup of Diff $_{+}^{2}(I)$ such that every nontrivial element of Γ has at most N fixed points. Then Γ is metaabelian.

An important tool in obtaining these results is provided by Theorems B-C from [A]. Theorem B (Theorem C) states that a non-solvable (non-metaabelian) subgroup of Diff $_{+}^{1+\epsilon}(I)$ (of Diff $_{+}^{2}(I)$) is non-discrete in the C_0 metric. Existence of C_0-small elements in a group provides effective tools in tackling the problem. Such tools are absent for less regular actions, and for the group $\mathrm{Homeo}_{+}(I)$ (even for $\mathrm{Diff}_{+}(I)$), the problem of characterizing subgroups where every non-identity element has at most $N \geq 2$ fixed points still remains open.

Basic Notations: Throughout this paper, G will denote the group Diff $_{+}^{1+\epsilon}(I)$ where $\epsilon \in (0, 1)$. We let $\Gamma \leq G$. For every $g \in \Gamma$, $Fix(g)$ will denote the set of fixed points of g in $(0,1)$. A fixed point $x_0 \in Fix(g)$ is called *tangential* if the function $h(x) = g(x) - x$ does not change its sign near x_0. For $f, g \in G$, we will say that (the graphs of) f and g have *a crossing* at $x_0 \in (0,1)$ if $f(x_0) = g(x_0)$; this crossing will be called tangential if $f(x) - g(x)$ does not change its sign near x_0. We will write $F(\Gamma) = \sup\limits_{g \in \Gamma \setminus \{1\}} |Fix(g)|$. (so $F(\Gamma)$ is either 0, or a positive integer, or infinity.) For $f \in G$, we will write

$$d(f) = \sup_{x \in [0,1]} |f(x) - x| \text{ and } s(f) = \inf_{x,y \in Fix(f) \cup \{0,1\}, x \neq y} |x - y|.$$

If $F(\Gamma) < \infty$, then we can introduce the following natural biorder in Γ: for $f, g \in \Gamma$, we write $g < f$ if $g(x) < f(x)$ near zero. If f is a positive element (w.r.t. the order) then we will also write $g << f$ if $g^n < f$ for every integer n; we will say that g is *infinitesimal* w.r.t. f. For $f \in \Gamma$, we write $\Gamma_f = \{\gamma \in \Gamma : \gamma << f\}$ (so Γ_f consists of diffeomorphisms which are infinitesimal w.r.t. f). Notice that if Γ is finitely generated with a fixed finite symmetric generating set, and f is the biggest generator, then Γ_f is a normal subgroup of Γ, moreover, Γ/Γ_f is Archimedean, hence abelian, and thus we also see that $[\Gamma, \Gamma] \leq \Gamma_f$.

For the rest of the paper we may and will assume that the action of Γ is irreducible, i.e. Γ has no global fixed point in $(0,1)$. All the used generating sets of groups will be assumed to be symmetric. Let us also clarify that we define a metaabelian group as a solvable group of derived length at most two (so, Abelian groups are special cases of metaabelian groups).

Acknowledgment. We are grateful to Andrés Navas for very useful conversations and for his valuable remarks on the manuscript. We are also grateful to the anonymous referee for reading the paper very carefully, and for valuable comments, corrections, and suggestions.

1. Preliminary Results

First, we will discuss the case of $N = 1$. Let us first show a quick proof of the fact that $N = 1 \Rightarrow$ *metaabelian*.

Proposition 1.1. If $F(\Gamma) = 1$ then Γ is metaabelian.

Proof. (following Farb-Franks, [FF]) If all finitely generated subgroups of a group are metaabelian then the group is metaabelian.

Hence we may assume that Γ is finitely generated with a fixed finite generating set.

Let f be the biggest generator of Γ. Let also $h \in \Gamma_f \backslash \{1\}$ such that h has at least one fixed point (if such h does not exists then Γ_f is Abelian, therefore Γ is metaabelian) and $h(x) > x$ near zero. We may also assume that Γ_f has no global fixed point. (if Γ_f has a global fixed point then it is Abelian, hence Γ is metaabelian.)

If f has no fixed point, then for sufficiently big n, fh^{-n} will have at least two fixed points, hence, contradiction.

Assume now f has one fixed point. Let a be the fixed point of h. By conjugating f by the element of Γ_f if necessary, we may assume that the fixed point of f is bigger than a. Then, again, for sufficiently big n, fh^{-n} will have at least two fixed points. $\square$

Notice that the above proof works also in Homeo $_+(I)$.

We now want to discuss the general case. We will make use of the following claim:

Proposition 1.2. Let Γ be non-solvable, $F(\Gamma) < \infty$, $\epsilon > 0$, and $x_0 \in (0,1)$. Then there exist $\gamma_1, \gamma_2 \in [\Gamma, \Gamma]$ such that $0 < \gamma_1(x_0) - x_0 < \epsilon$ and $0 < x_0 - \gamma_2(x_0) < \epsilon$.

Using this proposition, we immediately obtain the following lemma

Lemma 1.3. Let Γ be non-solvable, $F(\Gamma) < \infty$, $0 < a < b < 1$, $x_0 \in (0,1)$. Then there exists $\gamma \in [\Gamma, \Gamma]$ such that $\gamma(x_0) \in (a, b)$. $\square$

Proposition 1.2 follows from the following more general fact

Proposition 1.4. Let Γ be non-solvable, $F(\Gamma) < \infty$, $\epsilon > 0$, and $x_1, \ldots, x_n \in (0,1)$. Then there exists $\gamma \in [\Gamma, \Gamma]$ s.t. $|\gamma(x) - x| < \epsilon, \forall x \in [0,1]$ and $Fix(\gamma) \cap \{x_1, \ldots, x_n\} = \emptyset$.

Proof. Let $F(\Gamma) = N$. Let also $\eta \in \Gamma$ such that $\eta(x) < x$ near 1, and $maxFix(\eta) < min\{x_1, \ldots, x_n\}$ (such η exists by irreducibility of Γ). We will consider a finite family $\{\eta^k \ : \ -nN - 1 \leq k \leq nN + 1\}$ of diffeomorphisms. By uniform continuity, there exists $\delta > 0$ such that for all $x, y \in [0,1], -nN - 1 \leq k \leq nN + 1$, we have $|x - y| < \delta \Rightarrow |\eta^k(x) - \eta^k(y)| < \epsilon$.

By Theorem B in [A], there exists $f \in [\Gamma, \Gamma] \backslash \{1\}$ such that $d(f) < \delta$. (Theorem B, as stated in [A], claims the existence of such f in Γ, but it is immediately clear from the proof that f can be chosen from the subgroup $[\Gamma,\Gamma]$ [1]). Then $d(\eta^{-k}f\eta^k) < \epsilon$ for all $k \in \{-nN-1, -nN, \dots, n, nN+1\}$. Moreover, by pigeonhole principle, there exists $k \in \{1, \dots, nN, nN+1\}$ such that $\{x_1, \dots, x_n\} \cap Fix(\eta^{-k}f\eta^k) = \emptyset$. $\square$

We would like to emphasize that we do not know how to prove Proposition 1.2 directly, i.e. without using Proposition 1.4 which is significantly stronger.

Now we will prove a somewhat stronger claim which is more suitable for our purposes

Proposition 1.5. Let Γ be non-solvable, $F(\Gamma) < \infty$, $\epsilon > 0$, and $0 < a < b < 1$. Then there exists $\gamma \in [\Gamma,\Gamma]$ s.t. $|\gamma(x) - x| < \epsilon, \forall x \in [0,1]$ and $[a,b] \cap Fix(\gamma) = \emptyset$.

Proof. In the proof of Proposition 1.4, it is sufficient to choose $\eta \in \Gamma$ such that $\eta^{-1}(a) > b$. $\square$

Proposition 1.5. in its own turn implies the following stronger claim.

Lemma 1.6. Let Γ be non-solvable, $F(\Gamma) < \infty$, $0 < c < a < b < d < 1$, and $x_0 \in (c,d)$. Then there exists $\gamma \in [\Gamma,\Gamma]$ such that γ has no fixed point in (c,d) and $\gamma(x_0) \in [a,b]$.

Proof. If $x_0 \in [a,b]$ then the claim immediately follows from Proposition 1.5. Then, without loss of generality we may assume that $x_0 \in (c,a)$.

Let $\epsilon_0 \in (0, b-a)$,

$$A = \{f \in G \mid d(f) < \epsilon_0, Fix(f) \cap (c,d) = \emptyset, \text{ and } f(t) > t, \forall t \in (c,d)\},$$

and

$$S = \{x \in [c,d] \mid \exists n \geq 2 \text{ and } z_1, \dots, z_n \in (c,d), \gamma_1, \dots, \gamma_{n-1} \in A \text{ s.t.}$$

$$z_1 = x_0, z_n = x, \gamma_i(z_i) = z_{i+1}, \text{ and } |z_{i+1} - z_i| < \epsilon_0, 1 \leq i \leq n-1\}.$$

Notice that by Proposition 1.5 the set A is not empty.

If $S \cap (d,1) \neq \emptyset$ then we are done. Indeed, let $z \in S \cap (d,1)$. Then there exists $n \geq 2$ and $z_1, \dots, z_n \in (c,d), \gamma_1, \dots, \gamma_{n-1} \in A$ such that $z_1 = x_0, z_n = z$, and $|z_{i+1} - z_i| < \epsilon_0, \gamma_i(z_i) = z_{i+1}, 1 \leq i \leq n-1$. Then, since $\epsilon_0 < b - a$, there exists $m \in \{2, \dots, n\}$ such that $z_m \in (a,b)$. Now, we can take $\gamma = \gamma_{m-1} \dots \gamma_1$.

[1]The proof of Theorem B relies on the proof of Theorem A. Notice that, in the proof of Theorem A, the maps f and g belong to the commutator subgroup. Hence the map $h_1^{-1}h_2$ at the end of the proof belongs to the commutator subgroup as well.

If $S \cap (d, 1) = \emptyset$ then we have $y := \sup S \leq d$. But by Proposition 1.5. there exists $\omega \in G$ such that $\omega(x) > x$ for all $x \in [c, d]$ and $d(\omega) < \epsilon_0$. Then for sufficiently small $\delta > 0$, $\omega([y - \delta, y]) \subset (y, 1)$. But this contradicts the assumption that $y = \sup S$. $\square$

Remark 1.7. Let us notice that, in Propositions 1.4 and 1.5, Theorem B (from [A]) is used only to guarantee the existence of C_0-small elements. But if $\Gamma \leq \text{Diff}_+^2(I)$, then one can use Theorem C from [A] to guarantee the existence of C_0-small elements under a weaker condition that Γ is not metaabelian. Hence, if $\Gamma \leq \text{Diff}_+^2(I)$ then in Proposition 1.2, 1.4, 1.5 and Lemma 1.3, 1.6 one can replace the condition "Γ is not solvable" with "Γ is not metaabelian". Moreover, by Theorem B (see the statement in [A]), there exists $k(\epsilon)$ such that any C_0 discrete subgroup of G is solvable of derived length at most $k(\epsilon)$. Thus, for subgroup $\Gamma \leq G$, in Proposition 1.2, 1.4, 1.5 and Lemma 1.3, 1.6, the condition "Γ is not solvable" can be replaced with "the derived length of Γ is not bigger than $k(\epsilon)$". We will fix the number $k(\epsilon)$ for the rest of the paper.

2. Solvability of groups with small N

Our proof of the main theorem will use the assumption that $N \geq 5$. Therefore we need to take care of the cases $N \leq 4$ separately. It is also interesting to see how easy it is to prove Theorem 0.1 in the cases of $N \leq 4$, using the results of Section 1.

The cases of $N = 0, 1$ have been discussed earlier so it remains to study the cases of $N = 2, 3, 4$. Our proofs for these cases are somewhat different from each other, and the techniques used in these proofs will be useful for the reader as a preparation for the proof of the main theorem.

If $\gamma \in \Gamma, Fix(\gamma) = \{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$ then we will write $p_i(\gamma) = a_i$ for all $1 \leq i \leq n$. (so $p_i(\gamma)$ denotes the i-th fixed point of γ.)

Proposition 2.1. Let $\Gamma \leq G$ such that $F(\Gamma) \leq 4$. Then Γ is solvable.

Proof. Notice that if, in a group, all finitely generated subgroups are solvable with derived length not bigger than some universal $d \geq 1$ then the group is solvable of derived length at most d. Also, by Theorem B in [A] if $\Gamma \leq G$ is C_0 discrete then Γ is solvable, and the derived length of Γ is not bigger than $k(\epsilon)$. We may and will assume that $k(\epsilon) \geq 5$.

Keeping this and Remark 1.7 (the fact that in Propositions 1.4 and 1.5, Theorem B from [A] is used only to guarantee the existence of C_0-small elements) in mind, we will first assume that Γ is finitely generated with a fixed finite generating set.

Let $F(\Gamma) = N$, and $N \in \{2, 3, 4\}$. Let also f be the biggest generator of Γ.

Then $f \notin \Gamma_\gamma$ for all $\gamma \in \Gamma$ (in other words, f is not infinitesimal w.r.t. to any element of Γ), and $[\Gamma, \Gamma] \leq \Gamma_f$.

Without loss of generality, we may assume that f has no tangential fixed point. [Indeed, if f has a tangential fixed point then for sufficiently small $\delta > 0$ if $\omega \in [\Gamma, \Gamma]$ is such that $d(\omega) < \delta, Fix(\omega) \cap Fix(f) = \emptyset$ and $\omega(x) > x, \forall x \in (\delta, 1 - \delta)$ (such ω exists by Proposition 1.5), then there exists $\phi \in \{\omega f, \omega^{-1} f\}$ such that $|Fix(\phi)| \geq |Fix(f)|$, moreover, either ϕ has no tangential fixed point or it has more fixed points than f.[2] Notice also that ϕ is positive and $\phi \in \Gamma \backslash \Gamma_f$. Thus we can replace f by ϕ; and if we still have tangential fixed points we perform the same operation at most four times (since $F(\Gamma) = 4$) until we find a positive element in $\Gamma \backslash \Gamma_f$ (which we still denote by f) without tangential fixed points.]

Notice that $f(x) > x$ near zero. We may also assume that f has at least two fixed points. (From the proof of Proposition 1.1, we see that, for a suitable choice of $h \in \Gamma_f$, and for sufficiently big n, the

[2]It suffices to take

$$\delta = \frac{1}{2} \min\{|f(x + sr) - (x + sr)| \mid x \in Fix(f), s \in \{-1, 1\}\}$$

where $r = \frac{1}{4} \min\limits_{u,v \in Fix(f) \cup \{0,1\}, u \neq v} |u - v|$. To see why the claim holds, call a tangential fixed point $x \in Fix(f)$ *upper* *(lower)* if $f(z) \geq z$ $(f(z) \leq z)$ near x. Then, for all $x \in Fix(f)$,

i) if x is non-tangential, then f and ω will have at least one non-tangential crossing in $(x - r, x + r)$; the same holds also for f and ω^{-1};

ii) if x is upper-tangential, then f and ω will have at least two non-tangential crossings in $(x - r, x + r)$;

iii) if x is lower-tangential, then f and ω^{-1} will have at least two non-tangential crossings in $(x - r, x + r)$.

Thus if the number of upper-tangential fixed points of f is not less than the number of lower-tangential fixed points, then f and ω will have at least $|Fix(f)|$ non-tangential crossings; hence f and ω have either more than $|Fix(f)|$ crossings or exactly $|Fix(f)|$ non-tangential crossings. Similarly, if the number of lower tangential fixed points of f is not less than the number of upper tangential fixed points, then f and ω^{-1} will have at least $|Fix(f)|$ non-tangential crossings; hence f and ω^{-1} have either more than $|Fix(f)|$ crossings or exactly $|Fix(f)|$ non-tangential crossings.

diffeomorphism $h^{-n}f_1$ has at least two fixed points where $f_1 = \omega f \omega^{-1}$, for some $\omega \in \Gamma_f$. Hence, if necessary, we may replace f with $h^{-n}f_1$).

a) Let $N = 2$. Let also $h \in \Gamma_f \backslash \{1\}$ such that h has two fixed points (If such h does not exist then Γ_f is metaabelian by Solodov's Theorem, therefore Γ is solvable with derived length at most 3). We may also assume that h has no tangential fixed point, $Fix(h) \cap Fix(f) = \emptyset$, and $h(x) > x$ near zero.

Let $Fix(h) = \{a, b\}, a < b$. Let us remind that f has at least two fixed points, hence (since $N = 2$) exactly two fixed points.

Let c be the smallest fixed point of f. If necessary, using Lemma 1.3, by conjugating f by an element of Γ_f, we may assume that $c > b$. Then, for sufficiently big n, the graphs of f and h^n will have at least two crossings on the interval $(0, a)$, and at least one crossing on the interval (b, c). Hence $h^{-n}f$ will have at least three fixed points; contradiction.

Thus we proved that if Γ is finitely generated then it is solvable of derived length at most $\max\{3, k(\epsilon)\} = k(\epsilon)$. Now, to treat the case when Γ is not finitely generated, we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon)$, hence Γ itself is solvable of solvability degree at most $k(\epsilon)$.

b) $N = 3$. Again, we are assuming that Γ is finitely generated. Let $h \in \Gamma_f \backslash \{1\}$ such that h has three fixed points. (If such h does not exist then then by part a), Γ_f is solvable of solvability degree at most $k(\epsilon)$, therefore Γ is solvable of derived length at most $k(\epsilon)+1$). We may also assume that h has no tangential fixed points, $Fix(h) \cap Fix(f) = \emptyset$, and $h(x) > x$ near zero.

Let $Fix(h) = \{a, b, c\}, a < b < c$.

By assumption, f has at least two fixed points. By conjugating f if necessary, and using Lemma 1.3, we may assume that the smallest fixed point of f (let us denote it by p) lies in (b, c). Then, for sufficiently big $n \in \mathbb{N}$, the graphs of f and h^n have at least two crossings on $(0, a)$, and at least one crossing on (b, p). If f has three fixed points, then there will be one more crossings on $(p, 1)$. If f has two fixed points then there exists one more crossing either on (p, c) (if the biggest fixed point of f is less than c), or on $(c, 1)$ (if the biggest fixed point of f is bigger than c). Contradiction.

Now, to treat the case when Γ is not finitely generated we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon) + 1$. Hence Γ will be solvable of derived length at most $k(\epsilon) + 1$.

c) Let now $N = 4$. First, we are making the assumption that Γ is finitely generated. Again, there exists $h \in \Gamma_f \backslash \{1\}$ such that h has four fixed points. (If such h does not exist then by part b), Γ_f is solvable of solvability degree at most $k(\epsilon) + 1$, therefore Γ is solvable of derived length at most $k(\epsilon) + 2$). We may furthermore assume that h has no tangential fixed point, $Fix(h) \cap Fix(f) = \emptyset$, and $h(x) > x$ near zero.

By assumption, f has at least two fixed points. We may conjugate h to h_1 by elements of Γ_f such that $Fix(h_1) \subset (0, p_1(f))$. Then, for sufficiently big n, h_1^n and f have at least three crossings on $(0, p_1(f))$. Hence $f_1 = h_1^{-n} f$ has at least three fixed points. (Notice that f_1 is positive and $f_1 \notin \Gamma_f$.)

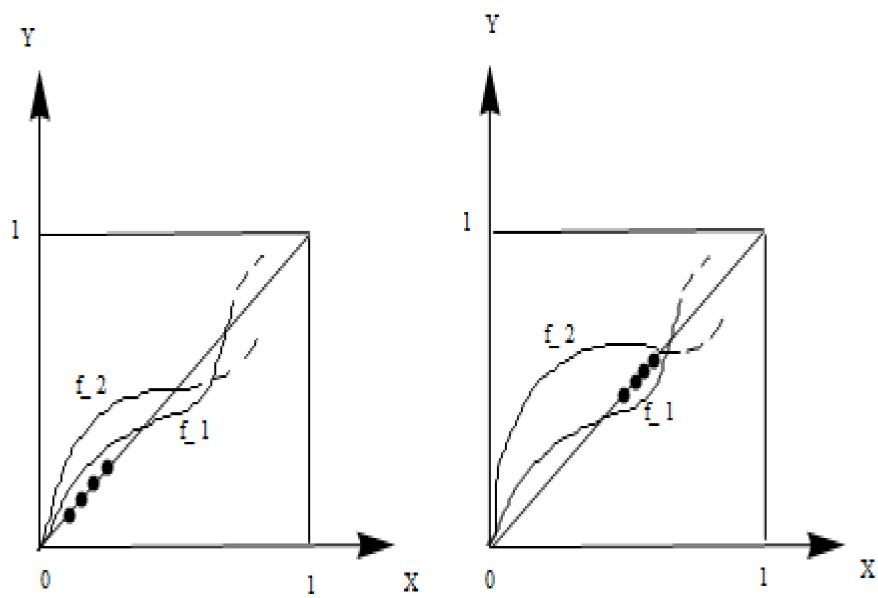

Figure 1a

Figure 1b

FIGURE 1. the four dots in Figure 1a (Figure 1b) represent the fixed points of h_2 (h_3).

We can conjugate f to f_2 by elements of Γ_f such that $p_1(f_2) \in (p_1(f_1), p_2(f_1))$. Then we can conjugate h to h_2 by elements of Γ_f such that $Fix(h_2) \subset (0, p_1(f_1))$ (Figure 1a). Then, conjugating h_2 by powers of f_2 we obtain h_3 where $Fix(h_3) \subset (p_1(f_1), p_1(f_2))$ (Figure 1b). Using Lemma 1.6, we can conjugate f_1 to f_3 by elements of Γ_f such that the first two fixed points of h_3 lie in $(0, p_1(f_3))$ while the last two fixed points lie in $(p_1(f_3), p_2(f_3))$(Figure 2).

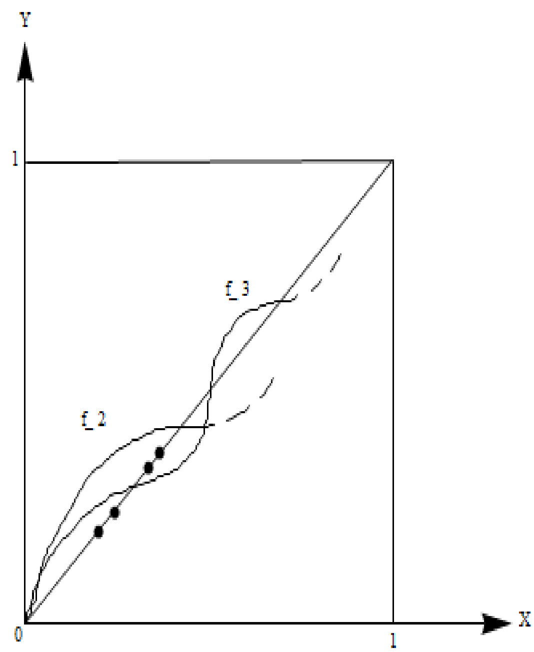

FIGURE 2. the four dots represent the fixed points of h_3.

Let $g_n = f_3^{-n} h_3 f_3^n, n \in \mathbb{N}$. Notice that as $n \to \infty$, we have $p_1(g_n) \to 0, p_2(g_n) \to 0$ while $p_3(g_n) \to p_2(f_3), p_4(g_n) \to p_2(f_3)$. Then for sufficiently big n, using Lemma 1.3, g_n can be conjugated to ϕ_n by an element of Γ_f such that ϕ_n has two fixed points in $(0, p_1(f_3))$ and another two fixed points in $(p_2(f_3), p_3(f_3))$. Then, for sufficiently big k, the diffeomorphisms ϕ_n^{-k} and f_3 have at least three crossings on $(0, p_1(f_3))$, and at least two crossings on $(p_2(f_3), p_3(f_3))$. Contradiction.

If Γ is not finitely generated then we notice that all its finitely generated subgroups are solvable of solvability degree at most $k(\epsilon) + 2$. Hence Γ is solvable of derived length at most $k(\epsilon) + 2$. $\square$

Let us emphasize that not only we proved that Γ is solvable but also the derived length of it is uniformly bounded above by $k(\epsilon)+2$.

Remark 2.2. Interestingly, already for $N = 5$, the elementary arguments in the proof of Proposition 2.1 do not seem to work, and one needs significantly new and more general ideas. (In fact, the value $N = 5$ seems to be too big for "elementary methods" and too small for "general methods". We make extra efforts to cover this particular case). The essence of the problem is that Lemma 1.3 can be viewed as a dynamical transitivity of the action. However, we do not have a result about dynamical k-transitivity even when $k = 2$. But as N gets bigger, we need higher transitivity results to control the picture as we did in the proof of Proposition 2.1. [*We call the action of* Γ *on* I k*-transitive if for all* $z_1, \dots, z_k \in (0,1)$*, where* $z_1 < z_2 < \dots < z_k$ *and for all open non-empty intervals* $I_1, I_2, \dots, I_k$ *where* $x < y$ *if* $x \in I_p, y \in I_q, 1 \leq p < q \leq k$*, there exists* $\gamma \in \Gamma$ *such that* $\gamma(z_i) \in I_i, 1 \leq i \leq k$*. Proposition 1.5 is indeed significantly stronger than 1-transitivity.*]

3. Proof of the Main Theorem

The following lemma is very interesting in itself. It will be used in the proof of Theorem 0.2, and the idea of the proof will be used repeatedly in the proof of Theorem 0.1.

Lemma 3.1 Let Γ be non-solvable, $F(\Gamma) < \infty, f \in \Gamma, |Fix(f)| = n$. Then there exists $\psi \in [\Gamma, \Gamma]$ such that $|Fix(\psi)| \geq n-1$. Moreover, ψ can be taken arbitrarily close (in the C_0-metric) to the identity.

Proof. Let $Fix(f) = \{c_1, \dots, c_n\}$ where $c_1 < \dots < c_n, \epsilon < \frac{1}{2}s(f), a = c_1 - \epsilon, b = c_n + \epsilon$. By uniform continuity of f and f^{-1}, there exists $\delta > 0$ such that if $d(h) < \delta$ then $d([f, h]) < \epsilon$. By Proposition 1.5, there exists $h \in [\Gamma, \Gamma]$ such that $h(x) > x$ on $[a, b]$, $Fix(h) \cap [a, b] = \emptyset$, and $d(h) < \min\{\delta, \epsilon\}$.

Let now $g = h^{-1}fh$. Then g has n fixed points $d_1, \dots, d_n$ where $d_1 < c_1 < d_2 < c_2 < \dots < d_n < c_n$. Then the element $\psi = f^{-1}g$ has at least $n-1$ fixed points $e_1, \dots, e_{n-1}$ where $e_i \in (c_i, d_{i+1}), 1 \leq i \leq n-1$. Moreover, since $d(h) < \delta$, we also have $d(\psi) < \epsilon$. $\square$

Now we want to observe the following simple lemma

Lemma 3.2. Let $f, h \in \text{Homeo}_+(I), Fix(h) \cap [a,b] = \{a,b\}, Fix(f) \cap [a,b] = \emptyset$. Let also $f(a) \in (a,b)$ or $f(b) \in (a,b)$. Then the graphs of $h^{-1}fh$ and f have at least one crossing on (a,b). $\square$

Remark 3.3. Under the assumptions of the lemma (indeed, under even a slightly more general assumption), f and h are called crossed elements. If f, h are crossed then the subgroup generated by f, h contains a non-abelian free semigroup([N]). (we do not use this result here).

In the proof of the main theorem, the case of $N = 5$ requires more effort; we will need the following lemma in treating this case.

Lemma 3.4. Let $\Gamma \leq G$ be a finitely generated subgroup with a fixed finite generating set and with derived length bigger than $k(\epsilon)$, $F(\Gamma) = 5$, and f be the biggest generator. Then at least one of the following conditions hold:

(i) there exists $g \in \Gamma \backslash \Gamma_f$ with exactly 5 fixed points.

(ii) $\Gamma \backslash \Gamma_f$ contains a diffeomorphism with exactly 4 fixed points and Γ_f does not contain a diffeomorphism with exactly 4 fixed points.

Proof. Since Γ is irreducible and Γ_f is a normal subgroup, we may assume that Γ_f has no global fixed point in $(0,1)$. As we already discussed, we may also assume that f has at least two non-tangential fixed points. Assume that there is no $g \in \Gamma \backslash \Gamma_f$ with 5 fixed points. Then there exists $h \in \Gamma_f \backslash \{1\}$ with exactly five non-tangential fixed points and $h(x) > x$ near zero. Using Lemma 1.3 (and Remark 1.7), we can conjugate h to h_1 by an element of Γ_f such that the biggest fixed point of h_1 lies in $(p_1(f), p_2(f))$. We can also conjugate h to h_2 such that $Fix(h_2) \subset (p_5(h_1), 1)$. [no need to use Lemma 1.3 here; we just use the fact Γ_f has no global fixed point in $(0,1)$]. Then, we can conjugate h_2 to h_3 by powers of h_1 such that $Fix(h_3) \subset (p_5(h_1), p_2(f))$. Finally, using Lemma 1.6, we can conjugate f to f_1 by an element of Γ_f such that the first two fixed points of h_3 lie in $(0, p_1(f_1))$ and the last three fixed points lie in $(p_1(f_1), p_2(f_1))$. Then, for sufficiently big n, the diffeomorphisms f_1 and h_3^n have at least three crossings in $(0, p_1(f_1))$, and at least one crossing in $(p_1(f_1), p_2(f_1))$. Thus, for sufficiently big n, the diffeomorphism $f_2 = h_3^{-n} f_1$ has at least four fixed points, hence exactly four fixed points. Notice that $f_2 \notin \Gamma_f$ and f_2 is positive.

Now, if there is no diffeomorphism in Γ_f with four fixed points then we are done. But if such a diffeomorphism exists then we fall in the case of the proof of Proposition 2.1, part c), (i.e. having a diffeomorphism $f_2 \in \Gamma \backslash \Gamma_f$ with at least three fixed points and a diffeomorphism Γ_f

with four fixed points) thus obtain a diffeomorphism with five fixed points in $\Gamma\backslash\Gamma_f$. $\square$

Now we are ready to prove the main result.

Proof of Theorem 0.1. We will assume that $N \geq 5$. Let $g \in \Gamma$ with N fixed points, and $g(x) > x$ near zero. We may assume that none of the fixed points of g is tangential. By conjugating g we obtain g_1 and g_2 such that $Fix(g_1) \subset (0, \frac{1}{10}), Fix(g_2) \subset (\frac{9}{10}, 1)$. Let $Fix(g_1) = \{a_1, \dots, a_N\}, Fix(g_2) = \{b_1, \dots, b_N\}$ where the elements are listed from the least to the biggest.

We will present a proof by contradiction. Using Proposition 1.5, as in the proof of Lemma 3.1, we obtain $h_1 \in \Gamma$ such that g_2 and $h_1^{-1}g_2h_1$ have at least one crossing in each of the intervals $(b_i, b_{i+1}), 1 \leq i \leq N-1$, moreover, if $d(h_1)$ is sufficiently small, then $d(f) < \min\{\frac{1}{10}, \frac{1}{2}s(g_1)\}$ where $f = g_2^{-1}h_1^{-1}g_2h_1$. Thus the diffeomorphism f has at least $N-1$ fixed points in $(b_1, 1) \subset (\frac{9}{10}, 1)$. Then f has at most one fixed point in $(0, \frac{9}{10})$.

Using Proposition 1.5 again, we obtain h_2 such that the diffeomorphisms g_1 and $h_2^{-1}g_1h_2$ have at least one crossing in each of the intervals $(a_i, a_{i+1}), 1 \leq i \leq N-1$, moreover, if $d(h_2)$ is sufficiently small, then $d(h) < \min\{\frac{1}{10}, \frac{1}{2}s(f)\}$ where $h = g_1^{-1}h_2^{-1}g_1h_2$. Thus the diffeomorphism h has at least $N-1$ fixed points in $(0, \frac{1}{10})$.

Let us now assume that $N \geq 9$. Then (let us recall that $d(f) < \min\{\frac{1}{10}, \frac{1}{2}s(g_1)\}$), if $d(h_2)$ is sufficiently small, there exist at least four mutually disjoint non-empty intervals $I_i = (p_i, q_i), 1 \leq i \leq 4$ in $(0, \frac{1}{5})$ such that for all $i \in \{1, 2, 3, 4\}$

(a1) $p_i, q_i \in Fix(h)$;

(a2) $Fix(h) \cap I_i = \emptyset$;

(a3) $f(p_i) \in I_i$ or $f(q_i) \in I_i$.

Since f has at most one fixed point in $(0, \frac{9}{10})$, f has no fixed points in at least three of these intervals. Thus, applying Lemma 3.2, we obtain that f and $h^{-1}fh$ have at least three crossings in $(0, \frac{9}{10})$. But since $d(h) < \frac{1}{2}s(f)$, we have at least one crossing in each of the intervals $(c_j, c_{j+1}), 1 \leq j \leq M-1$ where $c_1, \dots, c_M$ are all fixed points of f listed from least to the biggest, and $M \in \{N-1, N\}$ (we know that f has at least $N-1$ fixed points). Hence, f and $h^{-1}fh$ have at least $3 + (N-2) = N+1$ crossings. Contradiction.

When $N \in \{5,6,7,8\}$ we need to sharpen our observation. Indeed, for $N \in \{6,7,8\}$, we have either

Case 1. there exist four mutually disjoint non-empty intervals $I_i = (p_i, q_i), 1 \leq i \leq 4$ in $(0, \frac{1}{5})$ such that for all $i \in \{1,2,3,4\}$ the conditions (a1)-(a3) hold,

or

Case 2. there exist three mutually disjoint non-empty intervals $I_i = (p_i, q_i), 1 \leq i \leq 3$ in $(0, \frac{1}{5})$ such that for all $i \in \{1,2,3\}$ the conditions (a1)-(a3) hold, moreover, in two consecutive intervals, either $h(x) > x$ on both of these intervals, or $h(x) < x$.

In both cases, we obtain that f and $h^{-1}fh$ have at least three crossings in $(0, \frac{9}{10})$, hence, at least $N+1$ crossings total; contradiction.

The case of $N = 5$ needs a special treatment. We may assume that Γ is finitely generated with a fixed generating set and let θ be the biggest generator.

First, let us assume that $\Gamma \backslash \Gamma_\theta$ contains a diffeomorphism with 5 fixed points. The we choose g to be this diffeomorphism and we let g_1, g_2 be obtained from g by conjugating by elements from Γ_θ.

Then $f \in \Gamma_\theta$. Hence if f has a fixed point in $(0, b_1)$ then for sufficiently big n, f^n and g_2 will have at least two crossings in $(0, b_1)$, hence, at least $(N-1)+2$ crossings total, contradiction. Thus, we may assume that f has no fixed point in $(0, b_1) \supset (0, \frac{9}{10})$, and the conclusion follows as in the case of $N \geq 6$.

If $\Gamma \backslash \Gamma_\theta$ does not contain a diffeomorphism with 5 fixed points, then let H be an arbitrary finitely generated subgroup of Γ_θ with a fixed finite generating set, and let $\eta \in H$ be the biggest generator. Then, by Proposition 2.1, either $F(H) = 5$ or H is solvable of derived length at most $k(\epsilon)+2$. In the former case, if the derived length of H is bigger than $k(\epsilon)$, by Lemma 3.4, the subgroup H contains no diffeomorphism with four fixed points. Then, again by Lemma 3.4, we obtain that $H \backslash H_\eta$ contains a diffeomorphism with five fixed points. Thus applying the previous argument to this case we obtain a contradiction. Hence H is solvable of solvability degree at most $k(\epsilon)$.

Thus we obtained that an arbitrary finitely generated subgroup of Γ_θ is solvable of derived length at most $k(\epsilon)+2$. This implies that Γ_θ is solvable. Hence Γ is solvable. $\square$

For the purposes of clarity, let us emphasize that the proof of Theorem 0.1. does not use an inductive argument on $N \geq 5$.

Now we are ready to prove Theorem 0.2. Taking into account Remark 1.7, the proofs of Proposition 2.1. (for $N \in \{2,3,4\}$) and of Theorem 0.1. (for $N \geq 6$) go *mutatis mutandis* to obtain a contradiction under the assumptions that $\Gamma \leq \mathrm{Diff}\,^2_+(I)$ is non-metaabelian and $F(\Gamma) = N$, except we also need to show that the claims in parenthetical remarks in parts a)-c) of the proof of Proposition 2.1 hold as well. More precisely, we need to show that if $N \in \{2,3,4\}$ then Γ_f contains a diffeomorphism with N fixed points. Indeed, for $N = 2$, if Γ_f does not have such a diffeomorphism then by Theorem 1.5 of [FF], there exists an element of $h \in \mathrm{Homeo}_+(I)$ such that $h\Gamma_f h^{-1}$ is a subgroup of Aff_+I (this latter group, by definition, is obtained from $\mathrm{Aff}_+\mathbb{R}$ by conjugation by a tangent function). Then the subgroup $A = \{g \in h\Gamma_f h^{-1} \mid g = id \text{ or } Fix(g) = \emptyset\}$ contains elements arbitrarily close to the identity in the C_0-metric. Let $\phi \in h\Gamma h^{-1}$ with two fixed points. Then there exists $\omega \in A$ such that $[\omega, \phi] = \omega\phi\omega^{-1}\phi^{-1}$ has one fixed point. But $\phi\omega^{-1}\phi^{-1}$ has no fixed point, hence it belongs to A. Then $[\omega, \phi] \in A$, hence it has no fixed point; contradiction. For the case of $N = 3$, assume Γ_f has no element with three fixed points. If it also has no element with two fixed points then we run the same argument. But if $F(\Gamma_f) = 2$ then by part a), we obtain a diffeomorphism in Γ_f with three fixed points thus a contradiction. Similarly, for the case of $N = 4$: assume Γ_f has no element with 4 fixed points. If $F(\Gamma_f) = 1$ then we run the previous argument; but if $F(\Gamma_f) \in \{2,3\}$ then by parts a) and b) we obtain a contradiction again.

The case of $N = 5$ is again slightly subtle. In this case, we can only conclude that the group H (at the end of the proof of Theorem 0.1.) is metaabelian. Since H is an arbitrary finitely generated subgroup of Γ_θ then we can conclude that Γ_θ is metaabelian. This does not automatically imply that Γ is metaabelian (it only implies that Γ is solvable of derived length at most 3). So let us assume that Γ is not metaabelian.

If $\Gamma \backslash \Gamma_\theta$ contains a diffeomorphism with 5 fixed points then the claim again follows from the proof of Theorem 0.1; we choose g to be a diffeomorphism from $\Gamma \backslash \Gamma_\theta$ with five fixed points, and we let g_1, g_2 be obtained from g by conjugating by elements from Γ_θ.

If $\Gamma \backslash \Gamma_\theta$ does not contain a diffeomorphism with five fixed points then there exists $\xi \in \Gamma_\theta$ with five fixed points.

On the other hand, since Γ is not metaabelian, $[\Gamma, \Gamma]$ contains a non-trivial element η with $d(\eta) < \frac{1}{2}s(\xi)$ and $Fix(\eta) \cap Fix(\xi) = \emptyset$. Then

$Fix(\eta\xi\eta^{-1}) \cap Fix(\xi) = \emptyset$ and $\gamma = \eta\xi\eta^{-1}\xi^{-1}$ has at least four fixed points where $\gamma \neq 1$.

Since $[\Gamma, \Gamma] \leq \Gamma_\theta$ we have $\eta \in \Gamma_\theta$. But we also have $\xi \in \Gamma_\theta$ therefore $\gamma \in [\Gamma_\theta, \Gamma_\theta]$.

Now, since Γ is non-metaabelian, there exists $\omega \in [\Gamma, \Gamma] \leq \Gamma_\theta$ such that $d(\omega) < \frac{1}{2}s(\gamma)$ and $Fix(\omega) \cap Fix(\gamma) = \emptyset$. Then $Fix(\omega\gamma\omega^{-1}) \cap Fix(\gamma) = \emptyset$. Thus we obtain that $\omega\gamma\omega^{-1}$ and γ do not commute (otherwise γ would have to have infinitely many fixed points). But $\gamma \in [\Gamma_\theta, \Gamma_\theta]$, therefore $\omega\gamma\omega^{-1} \in [\Gamma_\theta, \Gamma_\theta]$. On the other hand, $[\Gamma_\theta, \Gamma_\theta]$ is Abelian. Then $[\gamma, \omega\gamma\omega^{-1}] = 1$. Contradiction. $\square$

References

[A] Akhmedov, A. A weak Zassenhaus lemma for subgroups of Diff(I). *Algebraic and Geometric Topology*, **14**, no.1, (2014), 539-550. http://arxiv.org/pdf/1211.1086.pdf

[B] Barbot, T. Characterization des flots d'Anosov en dimension 3 par leurs feuilletages faibles, *Ergodic Theory and Dynamical Systems* **15** (1995), no.2, 247-270.

[FF] Farb, B. and Franks, J. Groups of homeomorphisms of one-manifolds II: Extension of Hölder's Theorem. *Trans. Amer. Math. Soc.* **355** (2003) no.11, 4385-4396.

[K] Kovacevic, N. Möbius-like groups of homeomorphisms of the circle. *Trans. Amer. Math. Soc.* **351** (1999), no.12, 4791-4822.

[N] Navas, A. Groups of Circle Diffemorphisms. Chicago Lectures in Mathematics, 2011. *http://arxiv.org/pdf/0607481*

Azer Akhmedov, Department of Mathematics, North Dakota State University, Fargo, ND, 58108, USA

E-mail address: azer.akhmedov@ndsu.edu