

# EXISTENCE AND GENERICITY OF FINITE TOPOLOGICAL GENERATING SETS FOR HOMEOMORPHISM GROUPS

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ABSTRACT. We show that the topological groups  $\text{Diff}_+^1(I)$  and  $\text{Diff}_+^1(\mathbb{S}^1)$  of orientation-preserving  $C^1$ -diffeomorphisms of the interval and the circle, respectively, admit finitely generated dense subgroups. We also investigate the question of genericity (in the sense of Baire category) of such finite topological generating sets in related groups. We show that the generic pair of elements in the homeomorphism group  $\text{Homeo}_+(I)$  generate a dense subgroup of  $\text{Homeo}_+(I)$ . By contrast, if  $M$  is any compact connected manifold with boundary other than the interval, we observe that an open dense set of pairs from the associated boundary-fixing homeomorphism group  $\text{Homeo}_0(M, \partial M)$  will generate a discrete subgroup. We make similar observations for homeomorphism groups of manifolds without boundary including  $\mathbb{S}^1$ .

## 1. INTRODUCTION

A finite collection of elements  $g_1, g_2, \dots, g_n$  in a separable topological group  $G$  is called a (finite) *topological generating set* for  $G$  if the countable group  $\Gamma = \langle g_1, g_2, \dots, g_n \rangle$  which they generate is dense in  $G$ . In this case we say  $G$  is *topologically  $n$ -generated*. In 1990, Hofmann and Morris [6] showed that every separable compact connected group is topologically 2-generated. Their result built upon earlier work of Kuranishi [8] for compact semisimple Lie groups. Very recently, Geland and Le Maitre [4] have applied the solution to Hilbert's Fifth Problem to give a very general result: every separable connected locally compact group is topologically finitely generated.

The homeomorphism and diffeomorphism groups of compact manifolds are far from locally compact, and so different tools are needed for their study. Inspired by the programs of classical Lie theory, we are motivated in this paper by a desire to understand their discrete

and dense finitely generated subgroups<sup>1</sup>. In the recent article [11]<sup>2</sup>, Le Roux and Mann resolve the question of topological finite generation for a large class of diffeomorphism groups by studying a notion which they deem the Schreier property: a group has the Schreier property if each of its countable subsets is contained in some finitely-generated subgroup. Among their results they show that for a closed or open manifold  $M$ , the group  $\text{Diff}_0^k(M)$  of isotopically trivial  $C^k$  diffeomorphisms of  $M$  has the Schreier property whenever  $k \neq \dim M + 1$ . Since these groups are separable, it follows immediately that they admit dense finitely generated subgroups. However, the methods of [11] do not appear to apply directly to the case of compact manifolds with boundary; including the simplest case where  $M$  is the interval  $I = [0, 1]$ .

It is very well-known that the group  $\text{Homeo}_+(I)$  of orientation-preserving homeomorphisms of  $I$  is topologically 2-generated. The most popular example of a 2-generated dense subgroup in  $\text{Homeo}_+(I)$  is Thompson's group  $F$ , represented as a group of piecewise linear homeomorphisms. It is also well-known that the usual embedding of  $F$  may be "smoothed out" to find representations of  $F$  which lie in  $\text{Diff}_+^k(I)$ , for all  $1 \leq k \leq \infty$  [5]. However, these representations of  $F$  are never dense in  $\text{Diff}_+^k(I)$  with respect to the appropriate  $C^k$  topology. In fact, no finitely generated group has been shown to admit a  $C^1$ -dense representation in  $\text{Diff}_+^1(I)$ . Given that many separable topological groups do not admit finitely generated dense subgroups at all (this is easy to see in the additive product group  $\mathbb{Z}^{\mathbb{N}}$  for instance), we are motivated by the question: is  $\text{Diff}_+^1(I)$  topologically finitely generated or not? This question is especially intriguing since the topological group  $\text{Diff}_+^1(I)$  is homeomorphic to an infinite dimensional separable Banach space.<sup>3</sup>

The first main result of our paper is to show that, despite a lack of simple examples, dense finitely generated subgroups of  $\text{Diff}_+^1(I)$  do exist.

**Theorem 1.1.** *There exists a finitely generated dense subgroup of  $\text{Diff}_+^1(I)$ .*

The core of the proof lies in constructing a set of seven topological generators  $f, g, u, v, h, \phi, \psi$  for the commutator subgroup  $G$  of  $\text{Diff}_+^1(I)$ .

<sup>1</sup>The reader may consult the paper [1] of the first named author for a large number of remarks and open questions in this program.

<sup>2</sup>We thank the anonymous referee for calling our attention to this paper.

<sup>3</sup>Any two infinite dimensional separable Banach spaces are homeomorphic by a result of M. I. Kadets [9], and it is easy to see that the map  $f \rightarrow \ln f'(t) - \ln f'(0)$  establishes a homeomorphism from  $\text{Diff}_+^1(I)$  to the Banach space  $C_0[0, 1]$ .

Stated informally, we use four generators  $f, g, u, v$  to encode small copies of a countable dense subset  $(\eta_n)_{n \geq 1}$  of  $G$  along a sequence of disjoint subintervals  $(I_n)_{n \geq 1}$  of  $I$ . The fifth generator  $h$  is used to “erase” finitely many excess copies when they are not needed, while the sixth and seventh generators  $\phi$  and  $\psi$  are used to magnify the small copies to approximate the maps  $\eta_n$  to arbitrary precision.

Our construction relies on the perfectness results of Tsuboi [16] for diffeomorphism groups of the interval, as well as the following lemma on approximation by diffeomorphisms with iterative  $n$ -th roots.

**Lemma 1.2.** *Let  $f \in \text{Diff}_+^1(I)$  without a fixed point in  $(0,1)$  and assume that  $f'(0) = f'(1) = 1$ . Then for every  $r > 0$ , there exists a  $g \in \text{Diff}_+^1(I)$  and a positive integer  $N$  such that  $g$  is  $r$ -close to identity and  $g^N$  is  $r$ -close to  $f$ , in the  $C^1$  metric.*

While the preceding result is not new, we provide a direct, elementary, and self-contained proof in Section 3 for the reader’s interest.

Since  $\text{Diff}_+^1(I)$  admits a quotient group topologically isomorphic to  $\mathbb{R}^2$ , it is not too hard to see that a dense subgroup of  $\text{Diff}_+^k(I)$  requires at least 3 generators. Our construction uses 10 generators, and while this number should be easy to reduce, we are not sure of the least number of generators necessary. The least number of topological generators for a topological group  $G$  is sometimes called the *topological rank* of  $G$  (see [4]).

**Question 1.3.** What is the topological rank of  $\text{Diff}_+^1(I)$ ? Of  $\text{Diff}_+^k(I)$  for  $1 < k \leq \infty$ ?

In [4], the authors discuss the concept of *infinitesimal rank*. The infinitesimal rank of a separable topological group is the minimal number  $n$  such that, within any neighborhood of identity, there exists a set of  $n$  many topological generators for the group. If no such  $n$  exists, the infinitesimal rank of the group is said to be infinite. Examination of the proof of Theorem 1.1 reveals that the generators we construct may be chosen arbitrarily close to identity in the  $C^1$  metric, so the infinitesimal rank of  $\text{Diff}_+^1(I)$  is at most 10. In particular,  $\text{Diff}_+^1(I)$  has finite infinitesimal rank; and by using a standard fragmentation argument, so too does the diffeomorphism group  $\text{Diff}_+^1(\mathbb{S}^1)$  of the circle.

In the second part of this paper, we are concerned with the question of the generic behavior of finite topological generating sets. The question dates back to Schreier and Ulam ([13]), who showed that for every compact connected metrizable group  $G$ , the set of pairs  $(g_1, g_2) \in G$  which generate a dense subgroup of  $G$  has full Haar measure in  $G^2$ .

More recently, Winkelmann [17] has proved the following striking result in the case of connected Lie groups  $G$ : if  $G$  is amenable then there exists an  $N$  such that the set of  $n$ -tuples  $(g_1, \dots, g_n)$  which generate a dense subgroup has zero Haar measure for all  $n < N$ , while the set of  $n$ -tuples  $(g_1, \dots, g_n)$  which generate a discrete subgroup has zero Haar measure for all  $n \geq N$ . If  $G$  is non-amenable, then the set of discrete finitely generated groups and the set of non-discrete finitely generated groups both have infinite measure, for every  $n > 1$ .

In this paper we are primarily concerned with groups which are not locally compact and thus do not carry Haar measures. As the natural alternative, we approach the question in terms of Baire category. Recall that each group of the form  $\text{Homeo}_0(M)$  or  $\text{Diff}_0^k(M)$  for a compact manifold  $M$  is a Polish (i.e. separable completely metrizable) topological group and thus satisfies the hypothesis of the Baire category theorem (here,  $\text{Homeo}_0(M)$  denotes the connected component of identity in the full homeomorphism group  $\text{Homeo}(M)$ ; similarly,  $\text{Diff}_0^k(M)$  denotes the connected component of identity in the full diffeomorphism group  $\text{Diff}^k(M)$ ). We will say that a Polish topological group  $G$  is *generically topologically  $n$ -generated* if the set

$$\Omega_n = \{(g_1, g_2, \dots, g_n) \in G^n : \langle g_1, g_2, \dots, g_n \rangle \text{ is dense in } G\}$$

is comeager in  $G^n$  (i.e. if  $\Omega_n$  contains a countable intersection of dense open subsets of  $G^n$ ). If a group is generically topologically  $n$ -generated, then by the Kuratowski-Ulam theorem, it is generically topologically  $m$ -generated for all  $m \geq n$ .

As an example, each torus  $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$  is topologically 1-generated, so  $\Omega_1$  is non-empty. This is a consequence of a classical theorem of Leopold Kronecker ([7]), which asserts that  $\Omega_1$  of  $\mathbb{T}^n$  consists exactly of those tuples  $(g_1, \dots, g_n) \in \mathbb{T}^n$  for which  $\{1, g_1, \dots, g_n\}$  is a linearly independent set over the rationals. But this characterization implies something stronger, that  $\Omega_1$  is in fact comeager in  $\mathbb{T}^n$ . Similarly, one may apply Kronecker's theorem to show not only that each additive group  $\mathbb{R}^n$  is  $(n+1)$ -generated, but moreover that  $\Omega_{n+1}$  is comeager in  $\mathbb{R}^n$ . By contrast, Kechris and Rosendal ([10]) showed that the infinite permutation group  $S_\infty$ , endowed with its usual topology, is topologically 2-generated, but since  $S_\infty$  admits a base at identity of open subgroups,  $\Omega_n$  cannot be dense in  $S_\infty$ , let alone comeager. Rather, by direct argument it is easy to see that  $\Omega_n$  is nowhere dense in  $S_\infty$  for all  $n$ .

Our main result on this topic is the following.

**Theorem 1.4.** *Homeo<sub>+</sub>(I) is generically topologically 2-generated.*

It is striking that finitely generated dense subgroups are ubiquitous at the  $C^0$  level in  $\text{Homeo}_+(I)$ , and yet, speaking empirically, to find even one  $C^1$ -dense finitely generated subgroup in  $\text{Diff}_+^1(I)$  presents a great challenge. We remark that  $\text{Homeo}_+(I)$ , despite being non-locally compact, is in some ways analogous to a compact connected topological group, in the sense that it always has finite diameter with respect to any left-invariant metric on the group<sup>4</sup>, and thus has bounded large scale metric geometry. The diffeomorphism group  $\text{Diff}_+^1(I)$ , by contrast, has a very complicated and certainly unbounded large scale geometry. This fact may be reflected in the seeming paucity of finitely generated dense subgroups there.

In this paper we also point out that  $I$  is the unique compact connected manifold with boundary in which the generic finitely generated subgroup of homeomorphisms is dense. To state this more precisely, suppose  $M$  is a compact manifold with boundary, and denote by  $\text{Homeo}_0(M, \partial M)$  the connected component of the identity in the group of homeomorphisms of  $M$  which fix every point of the boundary  $\partial M$ . Note that if  $M = I$ , then  $\text{Homeo}_0(M, \partial M) = \text{Homeo}_+(I)$ , and  $\text{Homeo}_0(M, \partial M)$  is generically topologically 2-generated by our results. By contrast, if  $M \neq I$ , we have the following.

**Theorem 1.5.** *Let  $M$  be a compact connected manifold with boundary which is not homeomorphic to a closed interval. Then the set  $\Omega_n$  of all  $n$ -tuples which topologically generate  $\text{Homeo}_0(M, \partial M)$  is contained in a closed nowhere dense subset of  $(\text{Homeo}_0(M, \partial M))^n$  for all positive integers  $n$ .*

This theorem is shown using a standard ping-pong argument, which allows one to find open neighborhoods in a group whose elements always generate free discrete subgroups. The same argument shows that  $\text{Homeo}_+(\mathbb{S}^1)$  is not generically  $n$ -generated for any  $n$ , nor is  $\text{Homeo}_0(M)$  for any compact manifold  $M$  without boundary. This argument does not apply on the interval, which stands alone among the compact connected manifolds in this regard.

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<sup>4</sup>See [3], as well as [12] for exposition on the consequences of this property.

2. FINITENESS OF THE RANK OF  $\text{Diff}_+^1(I)$ 

We begin by showing that the topological commutator subgroup of  $\text{Diff}_+^1(I)$  is finitely generated. We use the following theorem, which appears as Theorem 4.1 in [16].

**Lemma 2.1** (Tsuboi). *The group  $G = \{f \in \text{Diff}_+^1(I) : f'(0) = f'(1) = 1\}$  is perfect.*

**Theorem 2.2.** *Let  $G = \{f \in \text{Diff}_+^1(I) : f'(0) = f'(1) = 1\}$ , endowed with the subgroup topology from  $\text{Diff}_+^1(I)$ . Then  $G$  is topologically 7-generated.*

**Proof.** By Lemma 2.1,  $G = \overline{[G, G]} = \overline{[\text{Diff}_+^1(I), \text{Diff}_+^1(I)]}$ . Let

$$S = \{f \in G \mid \text{Fix}(f) \cap (0, 1) = \emptyset\}.$$

We observe that  $S$  consists of exactly those maps satisfying the hypothesis of Lemma 1.2, and that  $S^2 = G$ .

We will prove that  $G$  is generated by seven diffeomorphisms  $\phi, \psi, f, g, u, v$  and  $h$  by constructing these diffeomorphisms explicitly.

Let  $I_n = (a_n, b_n)$ ,  $n \geq 0$  be open intervals in  $(0, 1)$  with mutually disjoint closures such that  $(a_n)_{n \geq 0}$  is decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ . Let also  $c_n \in (a_n, b_n)$ ,  $n \geq 0$ . Let  $(J_n)_{n \geq 0}$  be open mutually disjoint intervals in  $(0, 1)$  such that  $\overline{I_n} \subseteq J_n$ .

We choose  $\phi, \psi \in G$  such that the following conditions hold:

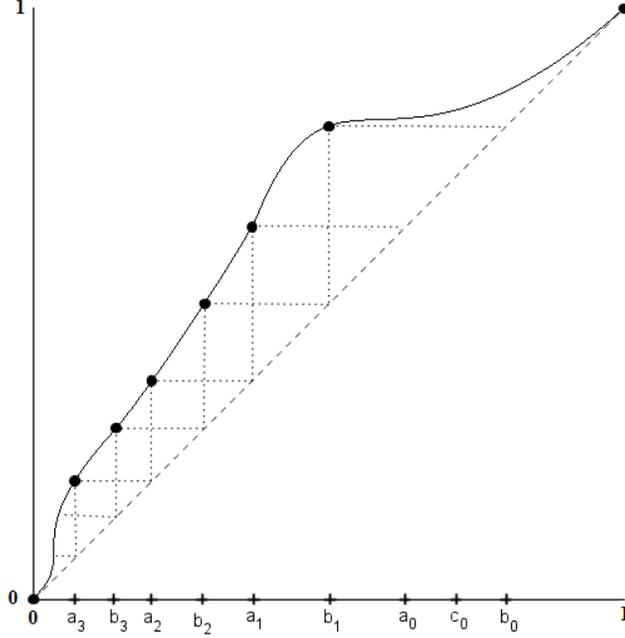
(a-i)  $\phi(x) > x, \forall x \in (0, 1)$ ;

(a-ii)  $\psi(c_0) = c_0$ , moreover,  $\psi(x) > x, \forall x \in (0, c_0)$  and  $\psi(x) < x, \forall x \in (c_0, 1)$ ; and

(a-iii)  $\phi^{-1}(a_n) = a_{n+1}, \phi^{-1}(b_n) = b_{n+1}, \phi^{-1}(c_n) = c_{n+1}, \forall n \geq 0$ .

For a natural number  $m$ , let us denote by  $\Phi_n^m$  the map  $\phi^n \psi^{-m} \phi^{-n}$ . In the construction that follows, for large values of  $m$ , the intuitive purpose of  $\Phi_n^m$  is to dilate maps (via conjugation) which are supported on  $[a_n, b_n]$ , to approximate maps which are supported on  $[0, 1]$ .

Now we start constructing  $f, g, u, v$  and  $h$ . For this, let  $\eta_1, \eta_2, \dots$  be a dense sequence in the subset  $\{[\omega_1 \omega_2, \omega_3 \omega_4] : \omega_i \in S, 1 \leq i \leq 4\}$  where we consider the standard  $C^1$  metric on  $G$ . Let also the sequence be enumerated in such a way that each map  $\eta_n$  is listed infinitely many times. Note that  $\langle \eta_n : n \geq 1 \rangle = \overline{[S^2, S^2]} = G$  where the latter equality follows from Lemma 2.1. Therefore it suffices for our construction to guarantee that each map  $\eta_n$  lies in the closure of the group  $\langle \phi, \psi, f, g, u, v, h \rangle$ .


 FIGURE 1. the map  $\phi$ .

We let  $f(x) = g(x) = u(x) = v(x) = x, \forall x \in I \setminus \bigsqcup_{n \geq 1} I_n$ , and  $h(x) = x, \forall x \in I \setminus \bigsqcup_{n \geq 1} J_n$ . We will complete the definition of  $f, g, u, v$  in the intervals  $I_n = (a_n, b_n), n \geq 1$ , and  $h$  in the intervals  $J_n$ , inductively. Let  $(\epsilon_n)_{n \geq 1}$  and  $(\delta_n)_{n \geq 1}$  be sequences of positive numbers decreasing to zero. Throughout the proof, if we state that a map  $f$  is  $\alpha$ -close to a map  $g$  on an interval  $J$ , we mean that  $f$  and  $g$  are uniformly  $\alpha$ -close in the  $C^1$ -metric on  $J$ , i.e.  $\sup_{x \in J} |f'(x) - g'(x)| < \alpha$ .

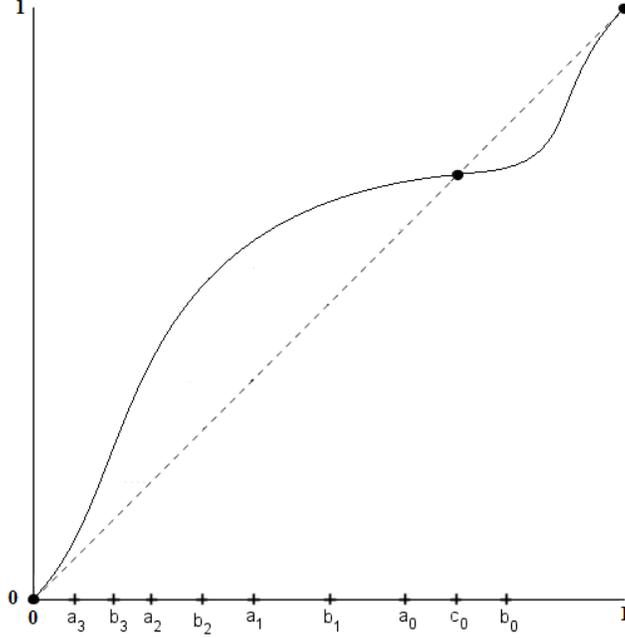
In the interval  $I_1$  we choose  $f, g, u, v$  so that:

(b-i)  $f'(a_1) = g'(a_1) = u'(a_1) = v'(a_1) = 1$  and  $f'(b_1) = g'(b_1) = u'(b_1) = v'(b_1) = 1$ ;

(b-ii)  $f, g, u, v$  are  $\delta_1$ -close to identity on the interval  $I_1$ ; and

(b-iii)  $\Phi_1^{m_1}[f^{M_1}g^{M_1}, u^{M_1}v^{M_1}](\Phi_1^{m_1})^{-1}$  is  $\epsilon_1$ -close to  $\eta_1$  on the interval  $\Phi_1^{m_1}(I_1)$  for some sufficiently big integers  $M_1$  and  $m_1$ .

We may achieve condition (b-iii) for sufficiently big  $M_1$  via Lemma 1.2. We define  $h$  so that  $h'(x) = 1$  if  $x$  is an endpoint of  $J_1$ ;  $h$  is  $\delta_1$ -close to identity on  $J_1$ ; and  $h(x) > x$  for all  $x \in J_1$ . Note that for some sufficiently large integer  $p_1$ , the supports of  $h^{p_1}fh^{-p_1}$  and  $h^{p_1}gh^{-p_1}$  in

FIGURE 2. the map  $\psi$ .

$J_1$  will each be disjoint from the supports of  $u$  and  $v$  in  $J_1$ . It follows that the commutator  $[h^{p_1} f^{M_1} g^{M_1} h^{-p_1}, u^{M_1} v^{M_1}]$  acts as the identity on  $J_1$ .

Suppose now the maps  $f, g, u$  and  $v$  are defined on  $\bigsqcup_{1 \leq i \leq n} I_i$  and  $h$  is defined on  $\bigsqcup_{1 \leq i \leq n} J_i$  so that

$$(c-i) \quad f'(x) = g'(x) = u'(x) = v'(x) = 1, \forall x \in \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\};$$

$$(c-ii) \quad f, g, u, v, h \text{ are } \delta_i\text{-close to identity on } J_i, 1 \leq i \leq n;$$

(c-iii) the map  $\Phi_n^{m_n} [h^{p_{n-1}} f^{M_n} g^{M_n} h^{-p_{n-1}}, u^{M_n} v^{M_n}] (\Phi_n^{m_n})^{-1}$  is  $\epsilon_n$ -close to  $\eta_n$  on the interval  $\Phi_n^{m_n}(I_n)$ , for some integers  $p_{n-1}, m_n, M_n$ ; and

(c-iv) for some integer  $p_n > p_{n-1}$ ,  $\Phi_1^{m_1} [h^{p_n} f^{M_1} g^{M_1} h^{-p_n}, u^{M_1} v^{M_1}] (\Phi_1^{m_1})^{-1}$  is the identity on  $\bigsqcup_{i \leq i \leq n} I_i$ .

Then we define the diffeomorphisms  $f, g, u, v$  on the interval  $I_{n+1}$  such that

$$(d-i) \quad f'(x) = g'(x) = u'(x) = v'(x) = 1 \text{ for all } x \in \{a_{n+1}, b_{n+1}\};$$

$$(d-ii) \quad f, g, u, v \text{ are } \delta_{n+1}\text{-close to the identity on } I_{n+1}; \text{ and}$$

(d-iii) the map  $\Phi_{n+1}^{m_{n+1}}[f^{M_{n+1}}g^{M_{n+1}}, u^{M_{n+1}}v^{M_{n+1}}](\Phi_{n+1}^{m_{n+1}})^{-1}$  is  $\epsilon_{n+1}$ -close to  $\eta_{n+1}$  on the interval  $\Phi_{n+1}^{m_{n+1}}(I_{n+1})$ , for some integers  $m_{n+1}, M_{n+1}$ .

We complete the inductive construction by choosing  $h$  on  $J_{n+1}$  in such a way that  $h$  is  $\delta_{n+1}$ -close to identity on  $J_{n+1}$  and also tangent to identity at the endpoints of  $J_{n+1}$ , and  $h(x) > x \forall x \in J_{n+1}$ . We also require  $h$  be so close to identity on  $J_{n+1}$  that the map

$$\Phi_{n+1}^{m_{n+1}}[h^{p_n} f^{M_{n+1}} g^{M_{n+1}} h^{-p_n}, u^{M_{n+1}} v^{M_{n+1}}](\Phi_{n+1}^{m_{n+1}})^{-1}$$

remains  $\epsilon_{n+1}$ -close to  $\eta_{n+1}$  on  $\Phi_{n+1}^{m_{n+1}}(I_{n+1})$ .

We note that for some sufficiently large integer  $p_{n+1} > p_n$ , the supports of  $h^{p_{n+1}} f h^{-p_{n+1}}$  and  $h^{p_{n+1}} g h^{-p_{n+1}}$  in  $J_{n+1}$  will each be disjoint from the supports of  $u$  and  $v$  in  $J_{n+1}$ , and thus the commutator

$$\Phi_{n+1}^{m_{n+1}}[h^{p_{n+1}} f^{M_{n+1}} g^{M_{n+1}} h^{-p_{n+1}}, u^{M_{n+1}} v^{M_{n+1}}](\Phi_{n+1}^{m_{n+1}})^{-1}$$

will act trivially on  $J_{n+1}$ , preserving the inductive hypothesis.

By induction, we extend the maps  $f, g, u, v, h$  to the whole interval  $I$ . Let  $X_n = X_n(\phi, \psi, f, g, u, v, h) = [h^{p_{n-1}} f^{M_n} g^{M_n} h^{-p_{n-1}}, u^{M_n} v^{M_n}]$ . By construction, the word  $\Phi_n^{m_n} X_n (\Phi_n^{m_n})^{-1}$  is:

1.  $\epsilon_n$ -close to  $\eta_n$  on the interval  $[\Phi_n^{m_n}(a_n), \Phi_n^{m_n}(b_n)]$ ;
2. the identity on  $[\Phi_n^{m_n}(b_n), 1]$ ; and
3.  $\gamma_n$ -close to identity on  $[0, \Phi_n^{m_n}(a_n)]$ , where  $\gamma_n$  depends only on  $\delta_n$  and  $m_n$ , and where  $\gamma_n \rightarrow 0$  as  $\delta_n \rightarrow 0$ .

By choosing  $(m_n)$  large enough we may guarantee that  $\eta_n$  is  $\epsilon_n$ -close to identity on the intervals  $[\Phi_n^{m_n}(b_n), 1]$  and  $[0, \Phi_n^{m_n}(a_n)]$ . Then, by choosing  $(\delta_n)$  small enough, we guarantee that  $\Phi_n^{m_n} X_n (\Phi_n^{m_n})^{-1}$  is  $\epsilon_n$ -close to  $\eta_n$  on  $I$ . Since each  $\eta_n$  is listed infinitely many times and  $\epsilon_n \rightarrow 0$ , this shows that  $\eta_n$  is in the closure of the group generated by  $\phi, \psi, f, g, u, v$  and  $h$ .  $\square$

We would like to describe the construction of the proof of Theorem 2.2 briefly in words. At each of the domains  $I_n$  the maps  $f, g, u, v, h$  generate some words  $X_n$ , each of which is a small copy of a map close to  $\eta_n$ . To dilate this small copy we use the maps  $\phi$  and  $\psi$ .  $\phi$  moves the domain  $I_n$  to  $I_0$  and  $\psi^m$  dilates the domain  $I_0$  to the interval containing  $(\alpha_n, 1 - \alpha_n)$  where  $\alpha_n$  tends to zero. The word  $X_n(f, g)$  has been realized as a commutator of a word  $Z_n(f, g, u, v)$  and the conjugate of a word  $Y_n(f, g, u, v)$  by some power of  $h$ . As a result of this we are able to make an arrangement that at all the intervals  $I_i, i < n$  conjugating by the chosen power of  $h$  disjoints the supports of  $Y_n(f, g, u, v)$  and  $Z_n(f, g, u, v)$  while  $h$  itself is very close to the identity in the  $C^1$  norm.

**Proof of Theorem 1.1.** We have shown that  $G$  is topologically generated by 7 maps  $\phi, \psi, f, g, u, v, h$ , and now as a consequence we show that  $\text{Diff}_+^1(I)$  is topologically 10-generated. We note that  $G$  is a closed normal subgroup in  $\text{Diff}_+^1(I)$ , and that the quotient  $\text{Diff}_+^1(I)/G$  is topologically isomorphic to the additive group  $\mathbb{R}^2$ . Since  $\mathbb{R}^2$  is topologically 3-generated, the claim follows from the following simple lemma; for the reader's convenience we present its proof.

**Lemma 2.3.** *Let  $H$  be a topological group and  $N$  be a normal subgroup. Then*

$$\text{topological rank}(H) \leq \text{topological rank}(H/N) + \text{topological rank}(N).$$

**Proof.** For  $x \in H$ ,  $[x]$  will denote the corresponding element in  $H/N$ . We may and will assume that the groups  $H/N$  and  $N$  have finite topological ranks. Let  $\text{topological rank}(H/N) = p$ ,  $\text{topological rank}(N) = q$ , and  $\langle [g_1], \dots, [g_p] \rangle = H/N$ ,  $\langle a_1, \dots, a_q \rangle = N$ .

We will prove that  $\langle g_1, \dots, g_p, a_1, \dots, a_q \rangle = H$ . Let  $g \in H$  be an arbitrary element, and  $U$  be an arbitrary non-empty open neighborhood of  $1 \in H$ . Consider the open set  $V := UN$  in  $H$ . Let  $V'$  be the projection of  $V$  in  $H/N$ . We can view  $V'$  as an open neighborhood of the identity in  $H/N$ . Then there exists a word  $w([g_1], \dots, [g_p])$  such that

$$[g]^{-1}w([g_1], \dots, [g_p]) \in V'.$$

This means that  $g^{-1}w(g_1, \dots, g_p) \in V = UN$ . Let  $g^{-1}w(g_1, \dots, g_p) = fn$ , where  $f \in U$ ,  $n \in N$ . Then there exists an open neighborhood  $A$  of the identity in  $N$  such that  $fA \subset U$ .

Now, we can find a word  $X(a_1, \dots, a_q)$  such that  $nX(a_1, \dots, a_q) \in A$ . Then  $g^{-1}w(g_1, \dots, g_p)X(a_1, \dots, a_q) \in U$ .

Since  $U$  is arbitrary, we are done.  $\square$

We remark that the analogue of Lemma 2.3 for infinitesimal rank is also true; see Lemma 4.5 in [4]. Thus  $\text{Diff}_+^1(I)$  is infinitesimally finitely generated.

### 3. ROOTS OF DIFFEOMORPHISMS

In this section we will prove Lemma 1.2 (represented here as Lemma 3.2). This lemma can also be proved combining the results of [14] and [15] about a germ of  $\text{Diff}_+^r(I)$  without fixed points being a time-one map of some  $C^1$ -vector field where  $r \geq 2$ , and then using the fact that  $\text{Diff}_+^r(I)$  is dense in  $\text{Diff}_+^1(I)$  in  $C^1$ -topology—see Section 1.4 of [2] for

an argument along these lines. For the reader's convenience, we present a proof that is direct, elementary, and totally self-contained. We will need the following.

**Definition 3.1.** Let  $l \geq 1$ . A finite sequence  $(t_i)_{1 \leq i \leq N}$  of real numbers is called *l-quasi-monotone* if there exists  $1 < i_1 \leq \dots \leq i_s < N$  for some  $s \in \{1, \dots, l\}$  such that each of the subsequences  $(t_{i_j}, \dots, t_{i_{j+1}})$  is monotone for every  $0 \leq j \leq s$ , where  $i_0 = 1, i_{s+1} = N$ . A 1-quasi-monotone sequence will be simply called *quasi-monotone*.

**Lemma 3.2.** Let  $f \in \text{Diff}_+(I)$ ,  $f'(0) = f'(1) = 1$  and  $r > 0$ . Let also  $f$  has no fixed point in  $(0, 1)$ . Then there exists a natural number  $N \geq 1$  and  $g \in \text{Diff}_+(I)$  such that  $\|g\|_1 < r$  and  $\|g^N - f\|_1 < r$ .

**Proof.** Let  $f_1 \in B_{r/4}(f)$  be a diffeomorphism of class  $C^2$  with no fixed point in  $(0, 1)$ , such that  $f_1''$  changes its sign at most  $l$  many times, for some finite  $l$ . Let also  $0 < x_0 < x_1 < \dots < x_n < 1$  such that  $f_1(x_i) = x_{i+1}, 0 \leq i \leq n-1$ , moreover,

$$\max\{x_0, 1 - x_n\} < r/4 \text{ and } \sup_{x \in [0, x_1] \cup [x_{n-1}, 1]} |f_1'(x) - 1| < r/4.$$

Let  $N \geq 1$  be sufficiently big with respect to  $\max\{n, l\}$ , and  $z_0, z_1, z_2, \dots, z_{nN}$  be a sequence such that  $z_{iN} = x_i, 0 \leq i \leq n$ ,  $z_k = x_0 + \frac{x_1 - x_0}{N}k, 0 \leq k \leq N$ , and  $z_{iN+j} = f_1(z_{iN-N+j}), 1 \leq i \leq n-1, 1 \leq j \leq N$ .

Now, we let

$$\bar{g}'(z_i) = \frac{z_{i+2} - z_{i+1}}{z_{i+1} - z_i} = \frac{g(z_{i+1}) - g(z_i)}{z_{i+1} - z_i}, 0 \leq i \leq nN - 2$$

and define the diffeomorphism  $g \in \text{Diff}_+(I)$  as follows. Firstly, we let  $g(z_i) = z_{i+1}, 0 \leq i \leq nN - 1$ . Then we let

$$g'(z_i) = \frac{z_{i+2} - z_{i+1}}{z_{i+1} - z_i} + \epsilon_i^{(N)}, 0 \leq i \leq nN - 2$$

where  $\epsilon_i^{(N)} < 0$  if  $\bar{g}'(z_i) < \bar{g}'(z_{i+1})$ , and  $\epsilon_i^{(N)} > 0$  if  $\bar{g}'(z_i) > \bar{g}'(z_{i+1})$ ; also, in case of the equality  $\bar{g}'(z_i) = \bar{g}'(z_{i+1})$ , we let  $\epsilon_i^{(N)} = 0$ .

Let us observe that we already have  $g^N(z_i) = z_{i+N}, 0 \leq i \leq nN - N$ . Then  $|g^N(z_i) - f_1(z_i)| < r/4$  for all  $1 \leq i \leq nN - 1$  hence we have  $\|g^N(x) - f_1(x)\|_0 < r/2$  **(1)** for all  $x \in [0, 1]$  (i.e. for all possible extensions of  $g$ ). On the other hand, by Mean Value Theorem, applied to the iterates  $f_1^s, 1 \leq s \leq n$  and their derivatives  $(f_1^s)', 1 \leq s \leq n$ , for

sufficiently big  $N$  and sufficiently small  $\epsilon^{(N)} := \max_{1 \leq i \leq N} |\epsilon_i^{(N)}|$ , we obtain that

$$|g'(z_i) - 1| < r/4, 0 \leq i \leq nN - 2.$$

By chain rule, we also have

$$(g^N)'(z_i) = \frac{z_{i+N+2} - z_{i+N+1}}{z_{i+1} - z_i} + \delta_i^{(N)}, 0 \leq i \leq nN - N$$

where  $\max_{1 \leq i \leq N} |\delta_i^{(N)}| \rightarrow 0$  as  $N \rightarrow \infty$  and  $\epsilon^{(N)} \rightarrow 0$ . Then by Mean Value Theorem, if  $N$  is sufficiently big, we obtain

$$|(g^N)'(z_i) - f_1'(z_i)| < r/4, 0 \leq i \leq nN - n.$$

Now we need to extend the definition of  $g'(x)$  to all  $x \in [0, 1]$  such that we have  $|(g^N)'(x) - f_1'(x)| < r/2$  **(2)** for all  $x \in [0, 1]$ . For all  $x \in (z_i, z_{i+1}), 0 \leq i \leq nN - 1$ , we define  $g'(x)$  such that  $g'(x)$  is monotone on the interval  $(z_i, z_{i+1})$ . (A monotone extension is possible because of the  $\epsilon_i^N$  error terms.)

Then we have  $|g'(x) - 1| < r/4$  for all  $x \in (x_0, x_n)$ . By uniform continuity of the derivative  $f_1'(x)$  and by Mean Value Theorem, if  $N$  is sufficiently big, we also obtain the inequality **(2)**. Indeed, it suffices to show that  $|(g^N)'(z) - (g^N)'(z_i)| < r/4$  for all  $z \in (z_i, z_{i+1}), 0 \leq i \leq nN - n$ .

By Mean Value Theorem, for sufficiently big  $N$  the sequence

$$(g'(z_i), g'(z_{i+1}), \dots, g'(z_{i+N}))$$

is  $l$ -quasi-monotone. For simplicity, let us assume that this sequence is quasi-monotone.

Without loss of generality, let us assume that  $g'(z_i) < g'(z_{i+1})$ . Then for some  $p \in \{1, \dots, N\}$  we have  $g'(z_i) \leq g'(z_{i+1}) \leq \dots \leq g'(z_{i+p})$  and  $g'(z_{i+p}) \geq \dots \geq g'(z_{i+N})$ .

Then

$$(g^N)'(z) \geq g'(z_i) \dots g'(z_{i+p-1}) g'(z_{i+p+1}) \dots g'(z_{i+N}) = g'(z_i) \dots g'(z_{i+N-1}) \frac{g'(z_{i+N})}{g'(z_{i+p})}.$$

For sufficiently big  $N$ , we obtain that  $(g^N)'(z) \geq (g^N)'(z_i) - r/4$ . Similarly, we also have the inequality  $(g^N)'(z) \leq (g^N)'(z_i) + r/4$ . Thus we obtain the inequality **(2)**.

If the sequence  $(g'(z_i), g'(z_{i+1}), \dots, g'(z_{i+N}))$  is  $l$ -quasi-monotone (i.e. in the most general case) then, similarly, we will obtain

$$(g^N)'(z) \geq g'(z_i) \dots g'(z_{i+N-1}) \prod_{1 \leq j \leq s} \frac{g'(z_{p_j})}{g'(z_{q_j})}$$

where  $s \leq l$  and  $p_1, q_1, \dots, p_s, q_s \in \{i, i+1, \dots, i+N\}$ . Then, by Mean Value Theorem, for sufficiently big  $N$ , we obtain the inequality  $(g^N)'(z) \geq (g^N)'(z_i) - r/4$ , and similarly, the inequality  $(g^N)'(z) \leq (g^N)'(z_i) + r/4$ .

The inequalities (1) and (2) imply the claim.  $\square$

#### 4. GENERIC SUBGROUPS

In this section we will prove that a generic 2-generated subgroup of  $\text{Homeo}_+(I)$  is dense. Moreover, we observe that  $I$  is the unique compact connected manifold whose homeomorphism group exhibits this property, for any size of finite generating set. That is, we prove that for any compact connected manifold  $M$  other than  $I$  and for any  $n \geq 1$ , a generic  $n$ -generated subgroup is not dense. In fact, we will exhibit an open set such that a subgroup generated by an  $n$ -tuple from this open set (in the product  $\text{Homeo}_0(M)^n$ ) is even discrete. If  $M$  is a compact manifold with boundary, then there is even an open dense set of generators in the product group  $\text{Homeo}_0(M)^n$  which generate a discrete subgroup. So the behavior of generic subgroups in a compact manifold  $M$  with boundary of dimension greater than or equal to 2 is opposite from the case of  $I$ .

**Theorem 4.1.** *For a comeager set of pairs  $(f, g) \in (\text{Homeo}_+(I))^2$ , the subgroup  $\Gamma = \langle f, g \rangle$  is dense in  $\text{Homeo}_+(I)$ .*

**Proof.** Let  $A$  be the set of all pairs  $(f, g)$  which generate a dense subgroup of  $\text{Homeo}_+(I)$ . We will show  $A$  is a dense  $G_\delta$  set in  $\text{Homeo}_+(I)$ . Let  $\rho$  denote the standard uniform metric on  $\text{Homeo}_+(I)$ . If  $D \subseteq \text{Homeo}_+(I)$  denotes a countable dense subset of  $\text{Homeo}_+(I)$ , and  $W$  denotes the set of all words in a free group on two generators, then we have

$$(f, g) \in A \leftrightarrow \forall d \in D \forall n \in \mathbb{Z}^+ \exists w \in W \rho(w(f, g), d) < \frac{1}{n}.$$

Since the predicate above is an open condition (by continuity of the word maps  $w$ ), this equivalence shows  $A$  is  $G_\delta$ . So it remains for us

to show that  $A$  is dense. Let  $(f, g)$  be an arbitrary pair of maps in  $\text{Homeo}_+(I)$  and let  $\delta > 0$  be arbitrary; we will show there is a pair of maps  $(\tilde{f}, \tilde{g})$  such that  $\rho(f, \tilde{f}) < \delta$ ,  $\rho(g, \tilde{g}) < \delta$ , and  $\Gamma = \langle \tilde{f}, \tilde{g} \rangle$  is dense.

Firstly, it is clear that the set of all pairs  $(f, g)$  which do not share a common fixed point in  $(0, 1)$  is dense in  $(\text{Homeo}_+(I))^2$ . This means we can assume without loss of generality that  $f$  and  $g$  do not fix a common point in  $(0, 1)$ , and we do so now.

We construct the maps  $\tilde{f}, \tilde{g}$  as follows. Since  $f$  and  $g$  fix the endpoints 0 and 1 of  $I$ , we may find a distance  $\alpha > 0$  so small that  $f(x), g(x) < \delta$  whenever  $x < \alpha$ . Let  $y_0 \in (0, \alpha)$  and  $\tilde{g}$  be any interval homeomorphism such that the following conditions are satisfied:

1.  $y_0$  is the least non-zero fixed point of  $\tilde{g}$ ;
2.  $\tilde{g}(x) > x$  for all  $x \in (0, y_0)$ ; and
3.  $\tilde{g}$  agrees with  $g$  on  $[\alpha, 1]$ .

Note that since we have only modified  $g$  on the interval  $[0, \alpha)$  to produce  $\tilde{g}$ , we have  $\rho(g, \tilde{g}) < \delta$  as desired.

Now let  $x_0 \in (0, y_0)$  be arbitrary. For each  $n \geq 1$  set  $x_n = \tilde{g}^{-n}(x_0)$ , so the sequence  $(x_n)$  decreases strictly and converges to 0. Let  $\phi$  and  $\psi$  be any two elements of  $\text{Homeo}_+([x_1, x_0])$  which generate a dense subgroup of  $\text{Homeo}_+([x_1, x_0])$ . Construct  $\tilde{f}$  to satisfy the following conditions:

1.  $\tilde{f}$  fixes each point in the sequence  $(x_n)$ ;
2.  $\tilde{f}(x) > x$  for all  $x \in (x_0, y_0]$ ;
3.  $\tilde{f}$  agrees with  $\phi$  on  $[x_1, x_0]$  and with  $\tilde{g}^{-1}\psi\tilde{g}$  on  $[x_2, x_1]$ ;
4.  $\tilde{f}$  agrees with  $\tilde{g}^{-2}\tilde{f}\tilde{g}^2$  on  $[x_{n+1}, x_n]$  for each  $n \geq 2$ ;
5.  $\tilde{f}$  agrees with  $f$  on  $[\alpha, 1]$ ; and
6.  $\tilde{f}$  has no common fixed point with  $\tilde{g}$  on  $(y_0, \alpha)$ .

Once again we have  $\rho(f, \tilde{f}) < \delta$  since we have only modified  $f$  on  $[0, \alpha)$ . Note that  $\tilde{f}$  is constructed in such a way that the maps  $\tilde{f}|_{[x_{n+1}, x_n]}$  and  $\tilde{g}\tilde{f}\tilde{g}^{-1}|_{[x_{n+1}, x_n]}$  generate a dense subgroup of  $\text{Homeo}_+([x_{n+1}, x_n])$  for each  $n \geq 0$ .

We claim that  $\tilde{f}$  and  $\tilde{g}$  generate a dense subgroup  $\Gamma = \langle \tilde{f}, \tilde{g} \rangle$  in  $\text{Homeo}_+(I)$ . To see this, let  $\varphi \in \text{Homeo}_+(I)$  and  $\epsilon > 0$  be arbitrary; we will produce a word in  $\tilde{f}$  and  $\tilde{g}$  which is uniformly  $\epsilon$ -close to  $\varphi$ .

Let  $\gamma > 0$  be so small that  $\varphi(x) < \epsilon$  whenever  $x < \gamma$  and  $\varphi(x) > 1 - \epsilon$  whenever  $x > 1 - \gamma$ . Note that the perturbed maps  $\tilde{f}$  and  $\tilde{g}$  still do not share a common fixed point in  $(0, 1)$ . This means we can find a map  $h \in \Gamma$  for which  $h(y_0) > 1 - \gamma$ . Set  $F = h\tilde{f}h^{-1}$  and  $G = h\tilde{g}h^{-1}$ . So  $F, G \in \Gamma$ , and  $\text{Fix}(F) = h(\text{Fix}(\tilde{f}))$  and  $\text{Fix}(G) = h(\text{Fix}(\tilde{g}))$ .

Since  $(h(x_n)) \rightarrow 0$ , let  $N$  be an integer so large that  $h(x_{N+1}) < \gamma$ . Since  $F$  has no fixed points on the interval  $(h(x_0), h(y_0)]$ , and the point  $G(h(x_0))$  lies on this interval, we may find an integer  $M$  so large that  $F^M(G(h(x_0))) > h(y_0)$ . Set  $\Phi = G^{-(N+1)}F^MG^{N+1} \in \Gamma$ . Then  $\Phi$  fixes  $h(x_{N+1})$ , and maps  $h(x_N)$  to a point strictly greater than  $h(y_0)$ . Defining  $a$  and  $b$  to be these image points respectively, we have

$$a = \Phi(h(x_{N+1})) = h(x_{N+1}) < \gamma < 1 - \gamma < h(y_0) < \Phi(h(x_{N+1})) = b.$$

Now since the maps  $\tilde{f}|_{[x_{N+1}, x_N]}$  and  $\tilde{g}\tilde{f}\tilde{g}^{-1}|_{[x_{N+1}, x_N]}$  generate a dense subgroup of  $\text{Homeo}_+([x_{N+1}, x_N])$ , it follows that the maps  $F|_{[h(x_{N+1}), h(x_N)]}$  and  $GFG^{-1}|_{[h(x_{N+1}), h(x_N)]}$  generate a dense subgroup of  $\text{Homeo}_+([h(x_{N+1}), h(x_N)])$ . In turn, we have that  $\Phi F\Phi^{-1}|_{[a, b]}$  and  $\Phi GFG^{-1}\Phi^{-1}|_{[a, b]}$  generate a dense subgroup of  $\text{Homeo}_+([a, b])$ .

Now note that  $|\varphi(a) - a| < \max(\varphi(a), a) < \epsilon$  since  $a < \gamma$ , and  $|\varphi(b) - b| < \max(1 - \varphi(b), 1 - b) < \epsilon$  since  $b > 1 - \gamma$ . This  $\epsilon$ -closeness at the endpoints  $a$  and  $b$  ensures that we may find a word  $w = w(\Phi F\Phi^{-1}, \Phi GFG^{-1}\Phi^{-1}) \in \Gamma$  which is uniformly  $\epsilon$ -close to  $\Phi$  on the interval  $[a, b]$ . Since this word  $w$  must fix  $a$  and  $b$  and  $a < \gamma < 1 - \gamma < b$ , it follows again from our choice of  $\gamma$  that  $w$  is uniformly  $\epsilon$ -close to  $\varphi$  on  $[0, a]$  and  $[b, 1]$  as well. So  $\rho(w, \varphi) < \epsilon$  and  $\Gamma$  is dense, as claimed.  $\square$

**Theorem 4.2.** *Let  $k \geq 0$  and  $M$  be a compact connected manifold which is not homeomorphic to a closed interval. Then for every  $n \geq 2$ , there exists  $f_1, f_2, \dots, f_n \in \text{Homeo}_0(M)$  and  $r > 0$  such that for all  $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n$  from the  $r$ -neighborhood of  $f_1, f_2, \dots, f_n$  in  $C^0$  metric, the maps  $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n$  generate a discrete free subgroup of rank  $n$  in  $C^0$  metric. Moreover, if  $M$  is a compact connected smooth manifold, then the same results hold for the groups  $\text{Diff}_0^k(M)$  in the  $C^k$  metric,  $1 \leq k \leq \infty$ , and we can choose  $f_1, f_2, \dots, f_n$  from  $\text{Diff}_0^\infty(M)$ .*

**Proof.** We will use a standard ping-pong argument. Let  $U$  be an open chart of the  $M$  homeomorphic to an open unit ball in  $\mathbb{R}^d$  where

$d = \dim M$ . Let also  $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$  be open balls in  $U$  with mutually disjoint closures such that  $\bigcup_{i=1}^n \overline{A_i \cup B_i} \subset U$ .

We can find homeomorphisms (or  $C^\infty$ -diffeomorphisms if  $M$  is a smooth manifold)  $f_1, \dots, f_n$  of  $M$  fixing  $M \setminus U$  such that for all  $1 \leq i \leq n$ ,

$$(i) f_i \left( \left( \bigcup_{\{j:1 \leq j \leq n, j \neq i\}} A_j \cup B_j \right) \cup \overline{A_i} \right) \subset A_i, \text{ and}$$

$$(ii) f_i^{-1} \left( \left( \bigcup_{\{j:1 \leq j \leq n, j \neq i\}} A_j \cup B_j \right) \cup \overline{B_i} \right) \subset B_i.$$

Then, for all non-zero integers  $m$ , for each  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , we have  $f_i^m(A_j \cup B_j) \subset A_i \cup B_i$ . Let  $\epsilon$  be the shortest distance among the balls  $A_1, \dots, A_n, B_1, \dots, B_n$ , and  $\delta > 0$  be such that  $\phi B_\delta(1) \phi^{-1} \subset B_\epsilon(1)$  for all  $\phi \in \{f_1^{\pm 1}, \dots, f_n^{\pm 1}\}$  (here,  $B_c(1)$  denotes the ball of radius  $c$  around the identity).

Then we can claim that for every non-trivial reduced word  $W$ , we have  $\|W(f_1, \dots, f_n)\|_0 > \delta$ . Hence  $\|W(f_1, \dots, f_n)\|_k > \delta$ .  $\square$

We remark here that although the previous theorem implies that  $\text{Homeo}_+(\mathbb{S}^1)$  is not generically topologically 2-generated, it does have dense 2-generated subgroups, and we provide a brief proof below.

**Proposition 4.3.**  *$\text{Homeo}_+(\mathbb{S}^1)$  is topologically 2-generated.*

**Proof.** Regard  $\mathbb{S}^1$  as the interval  $[0, 1]$  with the endpoints 0 and 1 identified. Let  $\phi : [0, 1] \rightarrow [0, 1/2]$  be any orientation-preserving homeomorphism. Let  $x_0$  and  $x_1$  denote the usual two generators of the usual representation of Thompson's group  $F$  inside  $\text{Homeo}_+(I)$ .

Let  $f = \phi x_1 \phi^{-1}$  and  $h = \phi x_1^{-1} x_0 \phi^{-1}$ , so  $f$  and  $h$  generate a dense subgroup of  $\text{Homeo}_+([0, 1/2])$ . Extend  $f$  and  $h$  to circle homeomorphisms by letting them act trivially on  $[1/2, 1]$ . Note that the map  $f$  is supported on  $[1/4, 1/2]$ , and  $f(x) < x$  for all  $x \in (1/4, 1/2)$ , while the map  $h$  is supported on  $[0, 3/8]$ , with  $h(x) < x$  for all  $x \in (0, 3/8)$ . It follows that  $f$  and  $h$  are conjugate in  $\text{Homeo}_+(\mathbb{S}^1)$ , by a fixed-point free homeomorphism  $g$  satisfying  $g(0) = 1/4$  and  $g(3/8) = 1/2$ . Let us also choose  $g$  in such a way that  $3/8$  is not a periodic point of  $g$ .

Let  $\Gamma = \langle f, g \rangle$ ; we claim  $\Gamma$  is dense in  $\text{Homeo}_+(\mathbb{S}^1)$ . For  $n \in \mathbb{N}$ , set  $a_n = g^n(0)$  and  $b_n = g^n(3/8)$ . Note that  $a_{n+1}$  lies between  $a_n$  and  $b_n$  for each  $n$ , so the intervals  $(a_n, b_n)$  and  $(a_{n+1}, b_{n+1})$  are overlapping. Since  $g$  is fixed-point free and  $b_0$  is not periodic, there exists an  $N$  for which  $b_N$  lies between  $a_0$  and  $b_0$ . Then the intervals  $\{(a_n, b_n)\}_{n=0}^N$  form an open cover of  $\mathbb{S}^1$ .

If  $k \in \text{Homeo}_+(\mathbb{S}^1)$  is arbitrary, then we may factorize  $k$  as  $k = k_0 k_1 k_2 \dots k_N$ , where the support of each  $k_n$  is a subset of  $[a_n, b_n]$ . Now by construction,  $\Gamma$  contains  $\langle f, g^{-1} f g \rangle = \langle f, h \rangle$  which is dense in  $\text{Homeo}_+[0, 1/2]$ . So  $k_0$  may be arbitrarily closely approximated by an element  $\gamma_0$  of  $\Gamma$ , since  $\text{supp}(k_0) \subseteq [a_0, b_0] \subseteq [0, 1/2]$ . Likewise, for each  $n \leq N$ , we have that  $\Gamma = g^n \Gamma g^{-n}$  is dense in  $\text{Homeo}_+[g^n(0), g^n(1/2)]$ , and  $\text{supp}(k_n) \subseteq [a_n, b_n] \subseteq [g^n(0), g^n(1/2)]$ . So  $k_n$  may be arbitrarily closely approximated by  $\gamma_n \in \Gamma$ . It follows that  $k$  is arbitrarily closely approximated by a product of the form  $\gamma_0 \gamma_1 \dots \gamma_N \in \Gamma$ . Since  $\bar{k}$  is an arbitrary element of  $\text{Homeo}_+(\mathbb{S}^1)$  to begin with, we see that  $\bar{\Gamma} = \text{Homeo}_+(\mathbb{S}^1)$ , i.e.  $\text{Homeo}_+(\mathbb{S}^1)$  is topologically generated by  $f$  and  $g$ .

**Theorem 4.4.** *Let  $M$  be a compact manifold with boundary such that no connected component of  $M$  is homeomorphic to a closed interval. Let  $\text{Homeo}_0(M, \partial M)$  denote the group of homeomorphisms of  $M$  which preserve the boundary  $\partial M$ . Then the set*

$$\{(f_1, f_2, \dots, f_n) \in (\text{Homeo}_0(M, \partial M))^n : \langle f_1, f_2, \dots, f_n \rangle \text{ is discrete}\}$$

*contains an open dense subset of  $(\text{Homeo}_0(M, \partial M))^n$ .*

**Proof.** Note that by hypothesis,  $\dim M \geq 2$ . Let  $(g_1, g_2, \dots, g_n) \in (\text{Homeo}_0(M, \partial M))^n$  and  $\epsilon > 0$  be arbitrary. Let  $x \in \partial M$  be a boundary point, and let  $\delta > 0$  be so small that  $|y - x| < \delta$  implies  $|g_i(y) - x| < \epsilon$  for all  $y \in M$ , for each  $1 \leq i \leq n$ . Let  $U$  be an open subset in the interior of  $M$  so that the closure of  $U$  is a subset of the  $\delta$ -ball about  $x$ . Now since  $M$  is at least a 2-dimensional manifold, we can perturb  $g_1, \dots, g_n$  to obtain new maps  $f_1, \dots, f_n$ , which agree with  $g_1, \dots, g_n$  respectively outside of the  $\delta$ -ball about  $x$ ; but which fix the boundary of  $U$ , and which exhibit the ping-pong behavior described in Theorem 4.2. Then  $f_1, \dots, f_n$  are  $\epsilon$ -close to  $g_1, \dots, g_n$  respectively, and any sufficiently small perturbations of the new maps  $f_1, \dots, f_n$  will generate a discrete subgroup of  $\text{Homeo}_0(M, \partial M)$ .  $\square$

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