My Favorite Things
Favorite Theorems, Objects, Concepts

azer.akhmedov

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I have selected the following items based on the fact that they have inspired me in my early studies of mathematics. Of course, as a mathematician, I still encounter facts and concepts that makes me very excited, but in my choices below, I have limited myself to my undergraduate years or earlier. For some of the listed items, even though I was (somewhat) familiar with them during my undergraduate years, I got to appreciate them better later.

I have heard (directly or indirectly) from several (well known) mathematicians that the Euler's Formula $e^{\pi i} = -1$ has been a call for them to mathematics, that they have been totally captured by the beauty and mystery of it. Maybe this is very strange, but I was never that much excited about this formula which to me is an indication that my list of favorite things below is probably very subjective.

1. (Cayley, 1889) There exist exactly n^{n-2} trees on n labeled vertices. The art of counting!

2. Every positive integer is a sum of four squares. Diophantus was aware of this theorem, but the proof was found by Lagrange only in 1770.

3. Pascal's Theorem: if a hexagon is inscribed in a conic then the intersection of opposite sides lie on the same line.

4. Classification of conics and quadrics

5. Any (smooth, complex and projective) cubic surface contains exactly 27 lines. Klebsh's Surface.

6. Borel sets. The concept of measure.

It took me years and even decades to really understand and appreciate that a measure, especially the one invariant under a group action, is an immensely powerful tool.

7. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges;

$\sum_{n=1}^{\infty} \frac{1}{n \log(n)}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n \log(n)^2}$ converges;

$\sum_{n=1}^{\infty} \frac{1}{n \log n \log\log(n)}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n \log n (\log\log(n))^2}$ converges;

and so on...

8. $$\sum_{p \text{ is prime}} \frac{1}{p} = \infty$$

9. Picard's Theorem. A non-constant entire function misses at most one value.

A much weaker version (Liouville's Theorem) is also very striking: a non-constant entire function is never bounded.

10. Fundamental Theorem of Algebra: algebraic, analytic and topological proofs.

11. Any connected Lie group of dimension 6 or less

12. Cardinality. Cantor's Theory.

13. Cantor set and its appearance/manifestation in so many places such as in ALexander's Horned Spheres and (as I was going to earn much later) in one-dimensional dynamics.

14. $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$

15. 3-dimensional Heisenberg group

16. Classification of simple Lie algebras over $\mathbb{C}$. Dynkin diagrams.

17. $\pi_k(X)$ is Abelian for all $k \geq 2$.

18. All classical functors of algebraic topology.
I was really crazy about these. I vividly remember avidly studying algebraic K-theory during which time I heard that there is also a J-functor. It had occurred to me J comes before K in the alphabet, "so shouldn't I study the J-theory first?", I was asking myself.

19. Long exact sequences. Especially the long exact homotopy sequence of a fibration.

20. Poincare duality (so all compact orientable manifolds of odd dimension have Euler characteristics zero).

21. Classification of 2-manifolds. Klein Bottle + $\mathbb{R}P^2 \cong \mathbb{T}^2 + \mathbb{R}P^2$.

22. Hopf Fibration (so $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$).

23. Harmonic functions. The fact that they are more smooth than one might think.

24. Homotopy groups of spheres.

25. Riemann's view of geometry: in the beginning there was a differential form.

26. Klein's view of geometry: in the beginning there was a group.

27. Whitney Embedding Theorem.

28. Massey Immersion Theorem.

As a graduate student, I impressed Massey once by quoting this theorem in my conversation with him at the department tea party. He told me (almost complained) that not all mathematicians are aware of this theorem.

29. Kodaira Embedding Theorem

I love all embedding theorems. Some others that I learned and loved a bit later: Sobolev Embedding Theorem, Nash Embedding Theorem.

30. Jensen's Inequality. In particular:

Harmonic Mean $\leq$ Geometric Mean $\leq$ Arithmetic Mean $\leq$ Quadratic Mean

31. Irrationality of e. Irrationality of π.

One number theorist told me that he is not interested in the irrationality of any number. I partially share this point of view, but at least the classic examples are exceptions.

32. Kuratowski Theorem (characterization of planar graph) and generalizations such as Seymour's Theorem.

33. Proof of 5-Color Theorem. Statement of 4-Color Theorem.

34. Tuttes Graph and all other counterexamples to "some regularity $\Rightarrow$ Hamiltonicity"

35. Banach-Alaoglu Theorem. Finally, the ball is compact!

36. Krein-Milman Theorem. Combining with Banach-Alaoglu Theorem to show that $C[0,1]$ is not a dual of any Banach space.

37. Bertrand's Postulate. Chebyshev's Proof.

38. Infinite dimensional Banach spaces in which strong and weak convergences coincide. l^1 is probably the best example.

39. The Law of Large Numbers. Kolmogorov's Theorems.

40. Cartan (KAK) and Iwasawa (KAN) Decompostions.

41. Main Theorem of Galois

42. Inverse Galois problems
I remember that, during the early years of graduate school, I had a rather rich collection of (types of) groups for which I knew how to realize as a Galois group over $\mathbb{Q}$.

43. Hilbert's Nullstellensatz

44. Coxeter Groups. Classification.

45. Archimedean Solids. Soccer ball is an Archimedean solid.

46. Existence and Uniqueness Theorems for ODEs
Take a map of a country, and lay a smaller map of the same country which totally lies inside a bigger one. Is there a point in the smaller map such that the

point right under it in the bigger map is geographically the same point? Such a point indeed exists (existence) and it is unique (uniqueness). For an ODE, the Existence and Uniqueness Theorem is essentially the same principle although in an infinite-dimensional setting.

47. Banach Algebras. Gelfand-Naimark Theorem.

I took a course in Banach algebras in my junior year - this was a mandatory course in former USSR for juniors. I was really inspired just by the definition and basic examples (just knowing that such things exist). In general, I always get excited by just putting any two mathematical structures together.

48. Understanding that, sometimes, despite of our effort of trying to define things abstractly (and broadly) they actually turn out to be very concrete. Examples: 1) every finite group is isomorphic to a subgroup of S_n for some n; 2) every separable infinite-dimensional Hilbert space is isomorphic to l_2.

49. Fractals. Mandelbrot set

50. Echman-Hilton Duality in algebraic topology

51. Eilenberg-MacLane spaces

52. Banach-Tarski Paradox

53. Hironaka's Theorem on Resolution of Singularities

54. Catalan Numbers. Knowing (if not 66, as Richard Stanley describes) but at least 10 different definitions.

55. Ramsey's Theorem
I was surprised to find that the following is just an application of Ramsey's Theorem: for any natural k, there exists a natural $N = N(k)$ such that given

any N points on the plane no three of which are collinear, some k of them will be vertices of a convex k-gon.

56. Zariski Topology

57. Fourier Transform

58. Lebesgue Density Theorem

It can be used in proving the following claim: For $A \subseteq \mathbb{R}^2$ and $r > 0$, the set $\{x \in \mathbb{R}^2 \mid d(x, A) = r\}$ has a zero measure.

59. Vitali Covering Lemma

60. Helly's Theorem

61. Computing the volume of the n-dimensional ball. Γ-function.

If v_n denotes the volume of the unit ball in $\mathbb{R}^n$, what is $\lim_{n\to\infty} v_n$? (v_n is also the ratio of the volume of the unit ball to the volume of the unit cube.)

62. Penrose Tiling

63. Groups of order 8. There are only two such non-Abelian groups up to an isomorphism, but they occur in many shapes such as $\mathbb{Z}_2 \wr \mathbb{Z}_2$, a unipotent group $U_3(\mathbb{F}_2)$, a dihedral group D_4, a quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ and so on.

64. Classifying groups of a given order at most 100. It is always fun (except perhaps the orders 32, 64, and 81). The case of order 16 is a well-studied non-trivial problem, so is the case of order 72.

65. Sylow Theorems. For what $n \leq 100$ do we have a finite simple non-

Abelian group of order n?

66. Actions of Braid group (in the case of the cyclic braid group $\mathbb{Z}$, this includes Fermat's Descent Method).

67. Tame knots, PL knots and differentiable knots define the same class up to an isotopy.

68. Number invariants (crossing number, genus, tunneling number, etc.) and polynomial invariants (Alexander, Kauffman, Jones) of knots.

69. Covering Theory

70. Elliptic Curves

71. Continued fractions

If you think you know this theory well here is an exercise to test yourself: Is it true that for all positive irrational numbers α, the image of the map $f : \mathbb{Z}^2 \to \mathbb{R}$ is dense where $f(x, y) = x^2 - \alpha y^2$?

72. Applications of Ergodic Theory in Number Theory.

This is by now a vast area with many celebrated results, but I was excited even about some very primitive applications. Just a dynamical reformulation of simple problems such as $\liminf\{\alpha n\} = 0$ was already quite attractive.

73. Birkhoff Convergence Theorem with all of its variations.

74. Riemann Integral and Lebesgue Integral. Studying these theories, using them and comparing them. In defining the first one, one chops the domain, and in the other one, the range.

75. Category Theory.

Like many undergraduates, I was also liking it to a degree that at some point categories were in my daily language. That's not the case anymore, but I still love category theory

76. Degree of a map. The degree of any map from $\mathbb{C}P^2$ to itself is a perfect square.

77. Foliations

78. Doing long computations in differential geometry.

79. Perhaps the two most favorite series:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \quad \text{and} \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

80. Hilbert's 3rd Problem.

81. Dirichlet Problem.

82. Sturm-Liouville Problem.

83. Interactions between topology and geometry. Too many important and bright examples to list (the solution of Poincare Conjecture being just one of them).

82. Mathematical Billiards. Is there a periodic billiard orbit on any triangle?

83. Schwartz theory of distributions. In particular, making Dirac's delta function rigorous; $\int_{-\infty}^{\infty} \delta(x)dx = 1,\ \delta * f = f.$

84. All kinds of elliptic-parabolic-hyperbolic classifications.

85. Predator-Prey Model.

86. Phase Transitions. Mathematical Theory of Phase Transitions. Ising Model and other mathematical models of statistical physics.

87. Quaternions.

88. Tangent Vector Fields. Tangent Bundles.

89. Parallelizable Manifolds. Parallelizable Spheres. Any kind of "only for $n = 1, 3, 7$" (or for $n = 1, 2, 4, 8$, or $n = 1, 2, 4, 8, 16$) type results.

90. Cobordisms. Thom's Theorem.

91. Haar Measure. Explicit computation of it for various Lie groups.
My first seminar talk as an undergraduate student was the existence and uniqueness result of a Haar measure in compact Lie groups. I love this proof which uses Kakutani Fixed Point Theorem.

92. Brower Fixed Point Theorem and all kinds of fixed point theorems including Lefcshetz FPT and Kakutani FPT.

93. A good PDE. Besides the heat, wave and Laplace equations, I also studied and loved Korteweg-de Vries equation.

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