Good modulating sequences for the ergodic Hilbert transform

Azer Akhmedov and Doğan Çömez [1]

Abstract. This article investigates classes of bounded sequences of complex numbers that are universally good for the ergodic Hilbert transform in L_p-spaces, $2 \le p \le \infty$. The class of bounded Besicovitch sequences satisfying a rate condition is among such sequence classes.

1. Introduction. Let (X, Σ, μ) be a probability space and $T : X \to X$ be an invertible measure preserving transformation. For any $f \in L_p$, the ergodic Hilbert transform (eHt) of f is defined as

$$Hf(x) := \lim_n \sum_{k=-n}^{n}{}' \frac{T^k f(x)}{k},$$

if the limit exists, where $\sum_{k=-n}^{n}{}'$ means summation without the term $k = 0$. It is well-known that the eHt exists a.e. for $f \in L_1$ [4, 11]. This result has also been extended to various other settings [1, 5, 8, 12, 6]. Given a sequence $\mathbf{a} = \{a_k\}$ of complex numbers, we will define the modulated ergodic Hilbert transform of $f \in L_p$ (modulated by $\mathbf{a}$) as $H_{\mathbf{a}} f(x) := \lim_n \sum_{k=-n}^{n}{}' \frac{a_k T^k f(x)}{k}$. If (X, Σ, μ, T) is a dynamical system, a sequence $\mathbf{a}$ is called *good for the ergodic Hilbert transform in* $L_p(X)$ if the modulated ergodic Hilbert transform exists μ-a.e. for every $f \in L_p(X)$. Let $\mathcal{T}$ be a class of measure preserving dynamical systems. We will say that the sequence $\mathbf{a}$ is *universally good for the ergodic Hilbert transform in* L_p *for the class* $\mathcal{T}$ if $\mathbf{a}$ is good for the ergodic Hilbert transform in L_p of every dynamical system in $\mathcal{T}$. In case that $\mathbf{a}$ is good for the ergodic Hilbert transform in L_p of *every* dynamical system, we will say that it is *universally good for the ergodic Hilbert transform in* L_p.

In the article [7] the second author investigated some classes of sequences that are universally good for the eHt. Such sequence classes are rather large; for instance, symmetric sequences of bounded variation and sequences of Fourier coefficients of functions in $L_p[0, 2\pi]$, $1 < p < \infty$, are universally good for the eHt. Recently, in [10] a Wiener-Wintner type theorem for the ergodic Hilbert transform was proved; a remarkable result which eluded mathematicians for two decades. A direct consequence of this result is that the sequences of the form $\{\lambda^k\}$, with $|\lambda| = 1$, are universally good for the eHt in L_p, $1 < p < \infty$. The techniques utilized in [7] fell short of proving that some sequence classes, known to be universally good for the ergodic averages, are universally good for the eHt. There, besides indicating that not all bounded Besicovitch sequences are good modulating for the eHt, it was proved that a proper subset of the set of bounded Besicovitch sequences is good for the eHt in $L_1(X)$. In this article, having the Wiener-Wintner theorem for the ergodic Hilbert transform, we will prove that those sequences are universally good for the eHt in L_2. Since we are in a probability space setting, these results also hold for L_p-functions for $2 \le p < \infty$. We also obtain other classes of sequences universally good for the eHt. Throughout this article, unless stated otherwise, we will assume $0 < \beta < 1$ and $1 < \alpha \le 2$. Also, $\mathbb{T}$ will denote the unit circle in complex plane, and C will always denote a constant, which may not be the same at each occurrence.

---

[1]* *2000 Mathematics Subject Classification.* Primary 20F16; Secondary 42B20, 40A30.

Key words and phrases. Hilbert transform, bounded Besicovitch sequences, modulating sequences.

2. Bounded universally good sequences for the eHt. In this section we will show that some fairly large classes of bounded complex sequences that satisfy a rate condition are universally good for the eHt. Let $\mathbf{a}$ be a sequence such that

$$(*)\quad \sum_{k=-n}^{n} |a_k| = O(n^\beta), \quad (n \geq 1),$$

In [7] it is shown that if $\mathbf{a}$ is a bounded sequence good for the ergodic theorem in L_∞ and satisfy the condition $(*)$, then it is universally good for the eHt in L_1. The class of sequences satisfying $(*)$ include the sequences of Fourier coefficients of functions belonging to the function spaces $L_p[0,2\pi]$, $1 < p < \infty$, $L_\alpha[0,2\pi]$ (the $\alpha-$Lipschitz functions in $L_1[0,2\pi]$), and $BV_1[0,2\pi]$ (the functions of bounded variation in $L_1[0,2\pi]$). The condition $(*)$ is naturally satisfied by the sequences belonging to these classes; however, the same assertions made there are also valid if one considers sequences satisfying a weaker condition. For, define

$$M_\alpha = \{\mathbf{a} : \sum_{k=-n}^{n} |a_k| = O\left(\frac{n^{\alpha-1}}{\log^\alpha n}\right)\}.$$

If n is large enough and $\alpha > \beta+1$, then $n^\beta \leq \frac{n^{\alpha-1}}{\log^\alpha n}$; hence, any sequence satisfying the condition $(*)$ belongs to M_α, if $\alpha > \beta + 1$.

In [2] it is shown that, among several other results related to the *one sided* ergodic Hilbert transform, if $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}}$ is a sequence of complex numbers satisfying

$$(**)\quad \sup_{n\geq 1} \max_{|z|=1} \frac{1}{n^{1-\beta}} |\sum_{k=1}^{n} a_k z^k| = C_{\mathbf{a}} < \infty,$$

then it is universally good for the eHt in L_1. When the (two-sided) ergodic Hilbert transform is concerned, it turns out that one can consider a larger class of sequences. Let A_α denote the set of sequences $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}}$ of complex numbers satisfying

$$(1)\quad \sup_{n\geq 1} \max_{|z|=1} \frac{\log^\alpha n}{n^{\alpha-1}} |\sum_{k=-n}^{n} a_k z^k| = C_{\mathbf{a}} < \infty.$$

Since $\frac{n^{\alpha-1}}{\log^\alpha n} \geq n^{1-\beta}$ when n is sufficiently large and $\alpha > 2-\beta$, all classes A_α contain sequences satisfying the condition $(**)$.

Remarks. 1. All the sequence classes mentioned above do not contain constant sequences; on the other hand, constant sequences are trivially universally good for the eHt [4, 11].

2. $M_\alpha \subset A_\alpha$. In fact, A_2 contains all M_α for all $1 < \alpha \leq 2$.

3. If $\mathbf{a} \in M_\alpha$, then for all $f \in L_\infty$, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f = 0$ a.e.

4. If $\alpha > 1+\beta$, then $\frac{n^{\alpha-1}}{\log^\alpha n} \geq n^\beta$. Hence it follows that any sequence satisfying the condition $(*)$ belongs to A_α. In particular, A_2 contains all such sequences.

5. There are A_α sequences that do not belong to any M_α. For instance, let $\mathbf{a} = \{a_n\}$ be the special case of the Hardy-Littlewood sequence given by $a_n = e^{in\log|n|}$. Clearly, $\mathbf{a} \notin M_\alpha$ since $\sum_{k=-n}^{n} |a_k| = O(n)$. However, for any $|z| = 1$, it follows that $|\sum_{k=-n}^{n} a_k z^k| = O(\sqrt{n})$ (see [13, p: 199]); hence $\mathbf{a} \in A_{3/2}$.

A sequence satisfying $(*)$(hence in M_α) need not be bounded. For example, let $a_k = j$ if $k = \mp 2^j$, and $a_k = 0$ for otherwise. Then $\mathbf{a}$ is an unbounded sequence and satisfies $(*)$ for any $\beta \in (0,1)$. Having this note, however, all the sequences considered throughout the rest of this article will be bounded.

Although sequences satisfying the condition $(*)$ are included in A_2, and $M_\alpha \subset A_\alpha$, for some values of α we also have the reverse inclusion.

Proposition 2.1 *Let $\alpha' + 1/2 < \alpha \leq 2$, then $A_{\alpha'} \subset M_\alpha$.*

Proof. By Hölder's inequality,

$$\sum_{k=-n}^{n} |a_k| \leq (2n)^{1/2} \left(\sum_{k=-n}^{n} |a_k|^2 \right)^{1/2} = (2n)^{1/2} \left[\int_{\mathbb{T}} |\sum_{k=-n}^{n} \lambda^k a_k|^2 d\lambda \right]^{1/2}.$$

Since $\mathbf{a} \in A_{\alpha'}$, we have $\sum_{k=-n}^{n} |a_k| \leq C \frac{n^{\alpha'-1/2}}{\log^{\alpha'} n}$ for some constant C. Hence, since $\alpha' + 1/2 < \alpha \leq 2$, the assertion follows. ∎

Remark. It follows from Proposition 2.1 that, if $\mathbf{a} \in A_\alpha$, $1 < \alpha < 3/2$, then $3/2 < 1/2 + \alpha < 2$, and hence, for all $f \in L_\infty$, $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} a_k T^k f = 0$ a.e.

Next, we will prove that M_α sequences are universally good for the eHt. The proof is essentially the same as the proof of Theorem 2.2 in [7]; hence, we will sketch it here for completeness.

Theorem 2.2 *Let $\mathbf{a} = \{a_k\} \in M_\alpha$. Then we have the weak (1,1) maximal inequality for $H_{\mathbf{a}} f$: for any $f \in L_1$, and for any $\lambda > 0$, there is a constant C such that*

$$\mu \left(\{x : \sup_{n\geq 1} |\sum_{k=-n}^{n}{}' \frac{a_k T^k f(x)}{k}| > \lambda\} \right) \leq \frac{C}{\lambda} \|f\|_1.$$

Furthermore, $\mathbf{a}$ is universally good for the eHt in L_1.

Proof. By Abel's summation by parts formula,

$$\sum_{k=-n}^{n}{}' \frac{a_k T^k f}{k} = \sum_{k=1}^{n-1} \frac{S_k - S_{-k}}{k(k+1)} + \frac{1}{n}(S_n - S_{-n}), \text{ where } S_{\mp j} = \sum_{i=1}^{j} a_{\mp i} T^{\mp i} f.$$

If $E = \{x : \sup_{n\geq 1} |\sum_{j=-n}^{n}{}' \frac{a_j T^j f(x)}{j}| > \lambda\}$, then $E \subset E_1 \cup E_2 \cup E_3$, where $E_1 = \{x : \sup_n |\frac{1}{n} S_n(x)| > \frac{\lambda}{3}\}$, $E_2 = \{x : \sup_n |\frac{1}{n} S_{-n}(x)| > \frac{\lambda}{3}\}$, and $E_3 = \{x : \sup_n |\sum_{j=1}^{n} \frac{1}{j(j+1)} [S_j(x) - S_{-j}(x)]| > \frac{\lambda}{3}\}$. Since we always have a weak (1,1) maximal inequality for the operators $\frac{1}{n} S_{\mp n}$ when $\mathbf{a}$ is a bounded sequence, $\mu(E_1) \leq \frac{C_1}{\lambda} \|f\|_1$ and $\mu(E_2) \leq \frac{C_2}{\lambda} \|f\|_1$, for some constants C_1 and C_2.

If $f \in L_1$ and $\mathbf{a} \in M_\alpha$, we have, for some constant C,

$$\int \left| \sum_{1\leq k\leq n} \frac{S_k - S_{-k}}{k(k+1)} \right| \leq \|f\|_1 \sum_{1\leq k\leq n} \frac{1}{k^2} \sum_{j=-k}^{k} |a_j| \leq \|f\|_1 \sum_{1\leq k\leq n} \frac{C}{k^{3-\alpha} \log^\alpha k} \leq C \|f\|_1,$$

where $C = \sum_{1\leq k\leq\infty} \frac{C}{k^{3-\alpha}\log^{\alpha} k}$. Since the sequence $\{h_n\} = \{\sum_{1\leq k\leq n} \frac{1}{k^2} \sum_{j=-k}^{k} |a_j|T^j|f|\} \subset L_1$ is monotone increasing with $\int h_n \leq C\|f\|_1$, by the Monotone Convergence Theorem, $\int h_n \uparrow \int h \leq C\|f\|_1$ where h is the pointwise limit of the sequence h_n. Hence, by Chebyshev's inequality, for any $\lambda > 0$, $\mu(E_3) \leq \frac{C}{\lambda}\|f\|_1$. Hence, the weak (1,1) maximal inequality for $H_{\mathbf{a}}f$ follows.

If $f \in L_\infty$ and $m < n$ are positive integers, then

$$\left|\sum_{m\leq k\leq n} \frac{S_k - S_{-k}}{k(k+1)}\right| \leq \|f\|_\infty \sum_{k=m}^{n} \frac{1}{k^2} \sum_{j=-k}^{k} |a_j| \leq \|f\|_\infty \sum_{k=m}^{n} \frac{C}{k^{3-\alpha}\log^{\alpha} k},$$

which implies that the sequence $\{\sum_{1\leq k\leq n} \frac{1}{k(k+1)}(S_k - S_{-k})(x)\}$ is Cauchy a.e.; hence, it converges. Since $\lim_n \frac{1}{n} S_{\mp n}(x)$ also converges a.e. for all $f \in L_\infty$, $\lim_n H_{\mathbf{a}}f(x)$ exists a.e. for all $f \in L_\infty$. By the Banach Principle, this fact combined with the weak (1,1) maximal inequality in the first part implies that $\mathbf{a}$ is universally good for the eHt in L_1. ∎

By Theorem 2.2 and Proposition 2.1, any $\mathbf{a} \in A_\alpha$ is also universally good for the eHt if $1 < \alpha \leq 3/2$. For $3/2 < \alpha \leq 2$ we need different arguments. Indeed, the statement below provides an argument valid for all $\mathbf{a} \in A_\alpha,\ 1 < \alpha \leq 2$.

Theorem 2.3 *If $\mathbf{a} = \{a_k\} \in A_\alpha,\ 1 < \alpha \leq 2$, is a sequence good for the ergodic averages, then it is universally good for the eHt in L_2.*

Proof. Since $\lim_n \frac{1}{n} S_{\mp n}$ exists a.e. by assumption, using Abel's partial summation, in order to show that $\lim_n \sum_{j=-n}^{n}{}' \frac{a_j T^j f(x)}{j}$ exists a.e. for all $f \in L_2$, all we need to show is that $\lim_n \sum_{j=1}^{n-1} \frac{S_j - S_{-j}}{j(j+1)}$ exists a.e., where $S_{\mp j} = \sum_{k=1}^{j} a_{\mp k} T^{\mp k} f(x)$.

Observe that, $\|S_j - S_{-j}\|_2^2 =< S_j, S_j > + < S_{-j}, S_{-j} > - < S_j, S_{-j} > - < S_{-j}, S_j >$, where $< f, g >= \int f\overline{g} d\mu$. Since

$$< S_j, S_j >= \sum_{k,l=1}^{j} < a_k T^k f, a_l T^l f >= \sum_{k,l=1}^{j} a_k \overline{a_l} < T^{k-l} f, f >,$$

by the spectral theorem for unitary operators, we have $< T^{k-l} f, f >= \int_{\mathbb{T}} z^{k-l} d\mu_f(z)$, where $\mathbb{T}$ is the unit circle. Therefore, $< S_j, S_j >= \int_{\mathbb{T}} \left[\sum_{k,l=1}^{j} (a_k z^k)(\overline{a_l z^l})\right] d\mu_f(z)$; and hence, it follows that

$$\|S_j - S_{-j}\|_2^2 = \int_{\mathbb{T}} \left[\sum_{k,l=1}^{j} (a_k z^k \overline{a_l z^l} - a_k z^k \overline{a_{-l} z^{-l}} - a_{-k} z^{-k} \overline{a_l z^l} + a_{-k} z^{-k} \overline{a_{-l} z^{-l}})\right] d\mu_f(z)$$
$$= \int_{\mathbb{T}} |\sum_{1\leq|k|\leq j} a_k z^k|^2 d\mu_f(z).$$

Since $\mathbf{a} \in A_\alpha$, it satisfies (1); hence, we have $|\sum_{-j}^{j} a_k z^k| \leq C_{\mathbf{a}} \dfrac{j^{\alpha-1}}{\log^\alpha j}$. Thus, $\|S_j - S_{-j}\|_2 \leq C\frac{j^{\alpha-1}}{\log^\alpha j}\|f\|_2$, for some constant C that depends on $\mathbf{a}$. Therefore, by Hölder's inequality, it follows that

$$\int \left|\sum_{j=1}^{n-1} \frac{S_j - S_{-j}}{j(j+1)}\right| \leq \int \sum_{j=1}^{n-1} \frac{|S_j - S_{-j}|}{j(j+1)} \leq C\|f\|_2 \sum_{j=1}^{n-1} \frac{1}{j^{3-\alpha}\log^\alpha j}.$$

Now, by the Monotone Convergence Theorem

$$\int \lim_n \left|\sum_{j=1}^{n-1} \frac{S_j - S_{-j}}{j(j+1)}\right| \leq \lim_n \int \sum_{j=1}^{n-1} \frac{|S_j - S_{-j}|}{j(j+1)} \leq C\|f\|_2 \sum_{j=1}^{\infty} \frac{1}{j^{3-\alpha}\log^\alpha j} < \infty;$$

hence, we deduce that $\lim_n \sum_{j=1}^{n-1} \dfrac{S_j - S_{-j}}{j(j+1)}$ exists a.e. ∎

Remark. It should be noted here that the arguments in the theorem above are purely L_2 space arguments.

The class of Besicovitch sequences are known to be universally good for the ergodic averages [2]. In [7] it has been proved that the sequences of Fourier coefficients of functions in $L_p[0, 2\pi]$, $1 < p \leq \infty$, which are bounded Besicovitch sequences, are universally good for the eHt in L_p, $1 < p \leq \infty$. There, it was also observed that *not every* sequence $\mathbf{a} \in B$ is good modulating for the eHt where B denotes the class of bounded Besicovitch sequences. On the other hand, a smaller subclass B_β of B produces good modulating sequences for the eHt, where

$$B_\beta = \{\mathbf{a} \in l_\infty : \exists\ \mathbf{w} \text{ induced by a trigonometric polynomial such that } \mathbf{a} - \mathbf{w} \text{ satisfies (*)}\}.$$

The techniques used in [7], however, fell short of showing that sequences in B_α are *universally* good for the eHt. In this section, having the Wiener-Wintner theorem for eHt [10], we will show that not only the sequence class B_α, but a subclass of bounded sequences that contains B_α provides sequences universally good for the eHt.

In [10], is was shown that if $f \in L_p$, $1 < p < \infty$, then there is a set $X_f \subset X$ of probability one such that for all $x \in X_f$

$$\lim_n {\sum_{-n}^{n}}' \frac{\lambda^k T^k f(x)}{k} \text{ exists for all } |\lambda| = 1.$$

Let $\mathcal{W}$ denote the class of sequences induced by bounded trigonometric polynomials, which are finite linear combinations of sequences of the form $\{\lambda^k\}$, $|\lambda| = 1$. Hence, it follows that the Lacey-Terwilleger Theorem holds for sequences in $\mathcal{W}$; that is, if $\mathbf{w} \in \mathcal{W}$ then it is universally good in L_p, $1 < p < \infty$.

Two-sided bounded Besicovitch sequences. First, we will consider two-sided bounded Besicovitch sequences $\mathbf{a} = \{a_k\}_{k\in\mathbb{Z}} \in B$, which are defined as, given $\epsilon > 0$, there exists $\mathbf{w}_\epsilon \in \mathcal{W}$ such that

$$(\dagger) \quad \limsup_n \frac{1}{n} \sum_{k=-n}^{n} |a_k - w_\epsilon(k)| < \epsilon.$$

Now, we define

$$MB_\alpha = \{\mathbf{a} \in l_\infty : \forall \epsilon > 0\ \exists\ \mathbf{w}_\epsilon \in \mathcal{W} \text{ such that } \limsup_n \frac{\log^\alpha n}{n^{\alpha-1}} \sum_{k=-n}^{n} |a_k - w_\epsilon(k)| < \epsilon\} \text{ and}$$

$$AB_\alpha = \{\mathbf{a} \in l_\infty : \exists\ \mathbf{w} \in \mathcal{W} \text{ such that } \mathbf{a} - \mathbf{w} \in A_\alpha\}.$$

Remarks. 1. $MB_\alpha \subset AB_\alpha$ for all α.

2. If $\mathbf{a} \in MB_\alpha$, then $\mathbf{a}$ is bounded Besicovitch.

3. If $\mathbf{a} \in B_\beta$, then $a_k = w_k + b_k$, where $\mathbf{w} = \{w_k\} \in \mathcal{W}$, and $\{b_k\}$ satisfies the condition $(*)$, hence $\mathbf{a} - \mathbf{w} \in M_\alpha$. Therefore, $B_\beta \subset MB_\alpha \subset AB_\alpha$ for all $0 < \beta < 1$ and $1 < \alpha \leq 2$.

4. Sequences induced by trigonometric polynomials belong to the sequence space MB_α; and hence, to AB_α.

By [10] and the remarks above, any $\mathbf{w} \in \mathcal{W}$ is universally good for the eHt in L_2. Since for $\mathbf{a} \in AB_\alpha$, $a_k = a_k - w_\epsilon(k) + w_\epsilon(k)$, where $\mathbf{w}_k$ is the appropriate trigonometric polynomial, all we need to prove is that $\left\{ {\sum_{-n}^{n}}' \frac{(a_k - w_\epsilon(k))T^k f}{k} \right\}_n$ converges a.e. In that case, using the same techniques as in Theorem 2.3, we obtain

Corollary 2.4 *If $\mathbf{a} \in AB_\alpha$, then it is universally good for the eHt in L_2.*

Remark. It follows from Corollary 2.4 that if $\mathbf{a} = \{a_k\} \in B_\beta,\ 0 < \beta < 1$, then it is universally good for the eHt in L_2.

Symmetric (one-sided) bounded Besicovitch sequences. In this section we will consider symmetric bounded Besicovitch sequences, namely, *ordinary* bounded Besicovitch sequences $\mathbf{a} = \{a_k\}$ (with $a_k = a_{-k}$) such that given $\epsilon > 0$, there exists $\mathbf{w}_\epsilon \in \mathcal{W}$ satisfying

$$\limsup_n \frac{1}{n} \sum_{k=1}^{n} |a_k - w_\epsilon(k)| < \epsilon.$$

First, we make an observation. Let $T : \mathbb{T} \to \mathbb{T}$ be an irrational rotation, say $Tz = \phi z$ for some $|\phi| = 1,\ \ \phi \neq 1$. Then for any λ on the unit circle, and for any f having ϕ as eigenvalue,

$$ {\sum_{k=-n}^{n}}' \frac{\lambda^{|k|} T^k f}{k} = f \sum_{k=1}^{n} \frac{(\phi\lambda)^k - (\bar{\phi}\lambda)^k}{k};$$

and hence, if $\lambda = \bar{\phi}$, then this series is not convergent. Therefore, symmetric bounded Besicovitch sequences defined by trigonometric polynomials need not be good for irrational rotations, which is different form the two-sided case.

Consider sequences $\mathbf{a} = \{a_k\}_{k \geq 0}$ such that $\gamma_{\mathbf{a}}(k) := \lim_n \frac{1}{n} \sum_{j=1}^{n} a_{j+k}\bar{a}_j$ exists for all $k \in \mathbb{N}$. $\gamma_{\mathbf{a}}$ is called the *correlation* of $\mathbf{a}$, which is extended to negative integers by letting $\gamma_{\mathbf{a}}(-k) = \bar{\gamma_{\mathbf{a}}}(k)$. Sequences $\{\gamma_{\mathbf{a}}(k)\}$ are positive definite; hence, by the Herglotz-Bochner theorem there exists a unique Borel probability measure $\mu_{\mathbf{a}}$ on the unit circle $\mathbb{T}$ such that

$$\gamma_{\mathbf{a}}(k) = \int_{\mathbb{T}} z^k d\mu_{\mathbf{a}}(z), \quad n \in \mathbb{N}.$$

The measure $\mu_{\mathbf{a}}$ is called the *spectral measure of* $\mathbf{a}$. Bounded Besicovitch sequences are known to have correlation; indeed, bounded Besicovitch sequences are exactly those complex sequences such that: (i) $\mu_{\mathbf{a}}$ is discrete, (ii) $\Gamma(z) := \lim_n \frac{1}{n}\sum_{j=0}^{n} a_j \bar{z^j}$ exists for every z, and (iii) $\mu_{\mathbf{a}}(z) = |\Gamma(z)|^2$ for all $z \in \mathbb{T}$ [2]. Furthermore, it is also known that $\Gamma(z) = 0$ for all but at most countably many $z \in \mathbb{T}$ [9]. The set $\sigma(\mathbf{a}) = \{z \in \mathbb{T} : \Gamma(z) \neq 0\}$ is called the *spectrum of* $\mathbf{a}$. Obviously, if $a_k = \lambda^k$ for some $\lambda \in \mathbb{T}$, then $\sigma(\mathbf{a}) = \{\lambda\}$.

If (X, Σ, μ, T) is an ergodic dynamical system, then $L_2(X) = \kappa \oplus \kappa^{\perp}$, where κ is the closed linear subspace spanned by the eigenfunctions of T (called the Kronecker factor of the system). Consequently, for a non-constant bounded Besicovitch sequence $\mathbf{a}$ and a measure preserving system with $\{f \in L_2 : Tf = f\} \subset \kappa$ properly, if $\lambda \in \sigma(\mathbf{a}) \cap \sigma(T)$, then, as observed above, $\lim_n \sum_{k=-n}^{n}{}' \frac{\lambda^{|k|} T^k f}{k}$ need not exist a.e. These arguments also imply that, given any dynamical system (X, Σ, μ, T) with $\{f \in L_2 : Tf = f\} \subset \kappa$ properly, there exists a bounded Besicovitch sequence $\mathbf{a}$ such that $\lim_n \sum_{k=-n}^{n}{}' \frac{a_{|k|} T^k f}{k}$ fails to exist a.e. However, as the next result shows, symmetric sequences $\mathbf{a} \in B_\alpha$ are still universally good in L_2 in a restricted sense. Let $\Im$ denote the class of weakly mixing measure preserving systems. If $T \in \Im$, then it has continuous spectrum, and hence, its Kronecker factor is simple; namely, $\kappa = \{f \in L_2 : Tf = f\}$. Therefore, for a weakly mixing T, $\lim_n \sum_{k=-n}^{n}{}' \frac{\lambda^{|k|} T^k f}{k}$ exists a.e. for any $f \in \kappa$. Again, using the same techniques as in Theorem 2.3, we obtain

Corollary 2.5 *If* $\mathbf{a} \in AB_\alpha$ *is a symmetric sequence, then it is universally good for the eHt in* L_2 *for the class* $\Im$.

The following theorem is analogous to Theorem 2.3, albeit with a restriction on the class of transformations. Let $\mathcal{L}$ denote the class of measure preserving dynamical systems having Lebesgue spectrum. Hence, the spectral measure of any nonconstant $f \in L_2$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$.

Theorem 2.6 *Let* $T \in \mathcal{L}$ *and* $\mathbf{a}$ *be a symmetric bounded sequence of complex numbers such that* $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} a_k T^k g(x)$ *exists a.e. for every* $g \in L_2$. *Then* $\mathbf{a}$ *is universally good for the eHt in* L_2 *for the class* $\mathcal{L}$.

Proof. First we observe that $L_2 = \mathbb{C} \oplus \mathcal{H}$, where $\mathcal{H} = \{g \in L_2 : \mu_g = hdz$ for some $0 \leq h \leq 1\}$. Since constants are trivially good for the eHt, it's enough to prove the assertion for $f \in \mathcal{H}$. Again, we will follow the proof of Theorem 2.3; and hence, it is enough to show that $\|S_n - S_{-n}\|_2 = O(n^\beta)$, for some $0 < \beta < 1$, where $S_{\mp j} = \sum_{i=1}^{j} a_i T^{\mp i} f$. For, using the spectral theorem, we obtain that

$$\|S_n - S_{-n}\|_2^2 \leq \int_{\mathbb{T}} \left| \sum_{k=1}^{n} (a_k z^k - a_k \bar{z}^k) \right|^2 d\mu_f(z),$$

and hence

$$\|S_n - S_{-n}\|_2 \le 2\left[\int_{\mathbb{T}} |\sum_{k=1}^{n} a_k z^k|^2 d\mu_f(z)\right]^{1/2} + 2\left[\int_{\mathbb{T}} |\sum_{k=1}^{n} a_k \bar{z}^k|^2 d\mu_f(z)\right]^{1/2}.$$

Since $\mathbf{a} \in l_\infty$, and T has continuous spectrum, we have,

$$\left[\int_{\mathbb{T}} |\sum_{k=1}^{n} a_k z^k|^2 d\mu_f(z)\right]^{1/2} \le \left[\int_{\mathbb{T}} |\sum_{k=1}^{n} a_k z^k|^2 dz\right]^{1/2} = \left[\|f\|_2^2 \sum_{k=1}^{n} |a_k|^2\right]^{1/2} = O(n^{1/2}\|f\|_2),$$

and similarly, $\left[\int_{\mathbb{T}} |\sum_{k=1}^{n} a_k \bar{z}^k|^2 d\mu_f(z)\right]^{1/2} = O(n^{1/2}\|f\|_2)$. This implies that $\|S_n - S_{-n}\|_2 = O(n^{1/2}\|f\|_2)$. ∎

At this point, one might ask if some other subclasses of bounded Besicovitch sequences are universally good for the eHt. One such candidate is the class of uniform sequences [3]. However, as the following example shows, they are not the right choice. Let $Y = \{1, 2, 3\}$, $\sigma : Y \to Y$ be a cyclic shift and μ be the uniform σ-invariant probability measure on Y (i.e., $\mu(\{i\}) = \frac{1}{3}$). Then (Y, μ, σ) is a strictly L-stable system; and hence, for each measurable $U \subset Y$ and $y \in Y$, the sequence $\{a_n\} = \{a_n(y, U)\}$ where

$$a_n = \begin{cases} 1 \text{ if } n \ge 0 \text{ and } \sigma^n(y) \in U, \\ -1 \text{ if } n < 0 \text{ and } a_{-n} = 1, \\ 0 \;\; \text{otherwise.} \end{cases}$$

Let $U = \{2\}$ and $y = 1$, then

$$a_n = \begin{cases} 1 \text{ if } n = 3m+1 \text{ and } m \ge 0, \\ -1 \text{ if } n = 3m-1 \text{ and } m \le 0, \\ 0 \;\; \text{otherwise.} \end{cases}$$

Now, given a measure preserving dynamical system (X, Σ, ν, τ), let X be divided into three sets A, τA and $\tau^2 A$ with each set having measure $\frac{1}{3}$. Define

$$f(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \in \tau A \\ -1 \text{ if } x \in \tau^2 A. \end{cases}$$

Then, for $x \in A$, we have,

$$\begin{aligned} f(\tau^{3k}x) &= 0, \quad f(\tau^{-3k}x) = 0 \\ f(\tau^{3k+1}x) &= 1, \quad f(\tau^{-3k-1}x) = -1 \\ f(\tau^{3k+2}x) &= -1, \quad f(\tau^{-3k-2}x) = 1. \end{aligned}$$

Therefore, $f \in L_2$ with $\|f\|_2 = \sqrt{\frac{2}{3}}$. Hence,

$$\sum_{i=-(3n+1)}^{3n+1} \frac{a_i f(\tau^i x)}{i} = \sum_{i=1}^{3n+1} \frac{a_i(f(\tau^i x) - f(\tau^{-i}x))}{i},$$

and for $x \in A$ we have,

$$\sum_{m=1}^{n} \frac{a_{3m+1}(f(\tau^{3m+1}x) - f(\tau^{-3m-1}x))}{3m+1} = \sum_{m=1}^{n} \frac{1-(-1)}{3m+1} = 2\sum_{m=1}^{n} \frac{1}{3m+1}.$$

Thus, $\sup_{n\geq 1}|\sum_{i=-(3n+1}^{3n+1}\frac{a_if(\tau^ix)}{i}|=\infty$; and hence, $\lim_n \sum_{i=-n}^{n}{}' \frac{a_if(\tau^ix)}{i}$ does not exist.

Remark. The system (X,Σ,ν,τ) is not weakly mixing, and the sequence $\{a_n\}$ has positive density.

Fourier coefficients of L_1 functions. In this section we would like to point out that, contrary to the L_p case ([7]), Fourier coefficients of a function $g\in L_1[0,2\pi]$ need not be universally good for the ergodic Hilbert transform. In fact a much stronger claim holds: there exists a function $g\in L_1[0,2\pi]$ such that for any dynamical system (X,Σ,μ,T), the Fourier coefficients of g is not good for the ergodic Hilbert transform in L_p for any $p\geq 1$.

We will make use of the fact that Fourier coefficients of the functions from $L_1[0,2\pi]$ may converge to zero arbitrarily slowly. Examples of such functions are well known (a somewhat implicit example can be found in [13], combining the results in Chapter V, p.183-184 and Chapter III, p.93.). We construct more direct, and somewhat different example (with a not necessarily convex sequence of Fourier coefficients).

Proposition 2.7. Let $h:\mathbb{N}\cup\{0\}\to\mathbb{R}$ be a function such that $\lim_{n\to\infty}h(n)=0$. Then there exists a sequence $\{a_n\}_{n\geq 0}$ such that $a_n\geq h(n)$ for every $n\in\mathbb{N}\cup\{0\}$, moreover, for all $x\in[0,2\pi]$, the series $\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n cos(nx)(\star)$ converges to $g(x)$, where $g\in L^1[0,2\pi]$, and $(\star)$ is the Fourier series of g.

For the proof of this proposition we will use the following common notations:

1. Given a sequence $\{x_n\}_{n\geq 0}$, we will write $\Delta x_n=x_n-x_{n+1}$, and $\Delta^2x_n=\Delta x_n-\Delta x_{n+1}$, for all $n\geq 0$.

2. For a function $g\in L^1[0,2\pi]$, we will write $g^+=g\chi_{\{x:g(x)\geq 0\}}$, and $g^-=g\chi_{\{x:g(x)\leq 0\}}$. So g^+ and g^- denote the positive part and the negative part of g respectively. Notice that $g(x)=g^+(x)+g^-(x)$, a.e. $x\in[0,2\pi]$.

3. For all $n\in\mathbb{N}, x\in[0,2\pi]$ we will write $D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}cos(kx)$ and $F_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}D_n(x)$.

Notice that for all $x\in(0,2\pi)$,

$$D_n(x)=\frac{sin(n+\frac{1}{2})x}{2sin(\frac{1}{2}x)} \text{ and } F_n(x)=\frac{2}{n+1}\left(\frac{sin\frac{1}{2}(n+\frac{1}{2})x}{2sin(\frac{1}{2}x)}\right)^2$$

Proof. Let $M=2\max\{h(n)\ :\ n\in\mathbb{N}\cup\{0\}\}$. Since $\lim_{n\to\infty}h(n)=0$, we can choose a piecewise linear function $a:[0,\infty)\to(0,\infty)$ satisfying the following conditions

(i) $a(n)>h(n),\forall n\in\mathbb{N}$;

(ii) a is strictly decreasing;

(iii) $a(0) = M$ and $\lim_{x\to\infty} a(x) = 0$;

(iv) a is differentiable on $(0,\infty)$ except at countably many points $n_1, n_2, \ldots$, where (assuming $n_0 = 0$), for all $k \in \mathbb{N}$, n_k is an integer and $n_k \geq n_{k-1} + 3$;

(v) there exists $\lambda > 1$ such that for all $k \in \mathbb{N} \cup \{0\}$, $n_{k+1} \leq \lambda(n_{k+1} - n_k)$;

(vi) if s_k denotes the slope of a on the interval (n_{k-1}, n_k) [i.e. $a'(x) = s_k, \forall x \in (n_{k-1}, n_k)$], then for all $k \in \mathbb{N}$, $s_k < s_{k+1} - s_k < -s_k$.

Then we define our sequence $\{a_n\}_{n\geq 0}$ by letting $a_n = a(n), \forall n \in \mathbb{N} \cup \{0\}$.

By definition of the sequence, if $n_{k-1} \leq n \leq n_k - 2$ then $\Delta^2 a_n = 0$, and if $n = n_k - 1$ then $|\Delta^2 a_n| = |s_k - s_{k+1}|$.

Then, by the conditions (i)-(vi), we obtain that $\sum_{n=1}^{\infty}(n+1)|\Delta^2 a_n| < \infty$ $(\star_1)$.

Indeed, by conditions (ii), (iii) and (iv), we have $\sum_{i=1}^{\infty}(-s_i)(n_i - n_{i-1}) = M$. Notice that $s_i < 0, \forall i \in \mathbb{N}$. Then, by condition (v), we obtain that $\sum_{i=1}^{\infty}(-s_i)n_i \leq \lambda \sum_{i=1}^{\infty}(-s_i)(n_i - n_{i-1}) < \infty$. This, combined with condition (vi), implies that $\sum_{i=1}^{\infty} |s_i - s_{i+1}| n_i < \infty$; thus we obtain $(\star_1)$.

Let now $s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k cos(kx)$ for all $n \geq 0, x \in [0, 2\pi]$. Since the sequence $\{a_n\}_{n\geq 0}$ is positive and decreasing, by Abel's summation formula, the limit $\lim_{n\to\infty} s_n(x)$ exists for all $x \in (0, 2\pi)$.

Also, if $x \in (0, 2\pi)$ and $n \geq 1$, by Abel's summation formula, we obtain that

$$s_n(x) = \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)$$

Applying Abel's summation formula again, for $n \geq 2$, we get

$$s_n(x) = \sum_{k=0}^{n-2}(k+1)\Delta^2 a_k F_k(x) + nF_{n-1}(x)\Delta a_{n-1} + a_n D_n(x)$$

Notice that $\lim_{n\to\infty} a_n D_n(x) = \lim_{n\to\infty} nF_{n-1}(x)\Delta a_{n-1} = 0$. Hence $g(x) = \lim_{n\to\infty} \sum_{k=0}^{n-2}(k+1)\Delta^2 a_k F_k(x)$.

Let us now show that g belongs to $L^1[0, 2\pi]$. For all $n \geq 2$, let

$$g_n(x) = \sum_{k=0}^{n-2}(k+1)\Delta^2 a_k F_k(x),$$

$$S_+(n) = \{k \ : \ 0 \leq k \leq n, \Delta^2 a_n > 0\}, S_-(n) = \{k \ : \ 0 \leq k \leq n, \Delta^2 a_k < 0\},$$

$$\phi_n(x) = \sum_{k\in S_+(n)} (k+1)\Delta^2 a_k F_k(x), \psi_n(x) = \sum_{k\in S_-(n)} (k+1)\Delta^2 a_k F_k(x)$$

Notice that $F_n(x) \geq 0, \forall x \in [0, 2\pi]$. Then for all $x \in [0, 2\pi]$ we have $g_n(x) = \phi_n(x) + \psi_n(x)$, where $\phi_n(x) \geq 0$ and $\psi_n(x) \leq 0$. Then $\phi_n(x) \geq g_n^+(x) \geq g_n(x), \forall x \in [0, 2\pi]$. By Fatou's Lemma, we have $\int_0^{2\pi} g^+ = \int_0^{2\pi} \lim_n g_n^+ \leq \int_0^{2\pi} \liminf_n \phi_n \leq \liminf_n \int_0^{2\pi} \phi_n$. On the other hand, $\int_0^{2\pi} F_k(x)dx = \pi$, for all $k \geq 1$. Then by the condition $(\star_1)$ we obtain that $\liminf_n \int_0^{2\pi} \phi_n < \infty$. Hence $g^+ \in L^1[0, 2\pi]$. Similarly, we obtain that $g^- \in L^1[0, 2\pi]$. Thus $g \in L^1[0, 2\pi]$.

After establishing integrability of g it follows from the claim **1-8** ([13], p.184]) that the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx)(\star)$ is indeed a Fourier series of g. $\square$

Let now (X, Σ, μ, T) be a dynamical system, and $g \in L^1[0, 2\pi]$ be a function with Fourier coefficients satisfying the following condition: $a_n = 0, \forall n \leq 0$, and $a_n \geq \frac{1}{\log(n)}, \forall n > 0$. Let also $\mathbf{a} = \{a_n\}_{n\in\mathbb{Z}}$ and $f(x) = 1, \forall x \in X$. Then $H_{\mathbf{a}} f(x) \geq \sum_{n=1}^{\infty} \frac{1}{n\log(n)} = \infty$, for all $x \in X$. Thus for any dynamical system (X, Σ, μ, T), the sequence $\{a_n\}$ is not good for eHt in L_p for any $p \geq 1$.

Remark. For convenience of the reader, we quote the claim **1-8** from [13]: *If* $\{a_n\}_{n\geq 1}$ *is a sequence of real numbers decreasing to zero and the function* $g(x) = \sum_{n=1}^{\infty} a_n cos(nx)$ *is integrable, then the series* $\sum_{n=1}^{\infty} a_n cos(nx)$ *is a Fourier series of* g.

Remark. It is indeed easy to construct a function $a(x)$ with the desired properties (i)-(vi). For all $k \in \mathbb{N}$, let $m_k = \min\{n \in \mathbb{N}\cup\{0\} : h(x) \leq \frac{M}{2^{k+1}}, \forall x \geq n\}$. Then let $\{n_k\}_{k\geq 0}$ be a sequence of non-negative integers such that $n_0 = 0, n_k \geq m_k$ and $n_{k+1} \geq 2n_k + 3$, for all $k \in \mathbb{N}\cup\{0\}$. We define the function $a(x)$ as follows: we let $a(n_k) = \frac{M}{2^k}, \forall k \in \mathbb{N}\cup\{0\}$, and we affinely extend the function to the interval $[n_k, n_{k+1}]$ for every $k \in \mathbb{N}\cup\{0\}$. Then, by taking $\lambda = 2$, it is clear that all of the conditions (i)-(vi) hold. Notice that the function $a(x)$ constructed in this way will be convex. In general though, conditions (i)-(vi) allow plenty of non-convex functions as well.

3. Extension to Admissible Processes. Given a sequence $\mathbf{a} \in l_\infty$, let

$$\|\mathbf{a}\|_\alpha := \limsup_{n\geq 1} \frac{\log^\alpha n}{n^{\alpha-1}} \sum_{-n}^{n} |a_k| < \infty.$$

Then $\| \|_\alpha$ defines a seminorm on M_α; that is $(M_\alpha, \| \|_\alpha)$ is a seminormed subspace of l_∞. Now, we turn to obtaining some properties of convergence with respect to $\| \|_\alpha$-seminorm, which will be instrumental in enlarging the scope of some family of good modulating sequences.

Definition. A sequence $\mathbf{a} = \{a_k\}_{k=-\infty}^{\infty}$ of complex numbers is called a *Hilbert sequence* if $\lim_n \sum_{k=-n}^{n}{}' \frac{1}{k} a_k$ exists.

Remark. For any $\lambda \in \mathbb{C},\ |\lambda| = 1$, the sequence $\{\lambda^k\}$ is a Hilbert sequence, and hence, every sequence induced by a trigonometric polynomial is a (bounded) Hilbert sequence.

Proposition 3.1 *a) If $\{\mathbf{a}^r\}$ is a Hilbert sequence for each $r \in \mathbb{Z}^+$ and if $\|\mathbf{a}^r - \mathbf{a}\|_\alpha \to 0$ as $r \to \infty$, then $\mathbf{a} = \{a_k\}$ is a Hilbert sequence.*

b) If $\mathbf{a}^r, \mathbf{a} \in M_\alpha$ with $\|\mathbf{a}^r - \mathbf{a}\|_\alpha \to 0$ and $\mathbf{a}^r\mathbf{b}$ is a bounded Hilbert sequence for all $r \in \mathbb{Z}^+$, for some $\mathbf{b} \in l_\infty$, then $\mathbf{ab}$ is a bounded Hilbert sequence.

Proof. Since

$$\sum_{k=-n}^{n}{}' \frac{1}{k} a_k = \sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r) + \sum_{k=-n}^{n}{}' \frac{1}{k} a_k^r,$$

and since $\{\sum_{k=-n}^{n}{}' \frac{1}{k} a_k^r\}_n$ converges, in order to prove (a) it is enough to show that $\{\sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r)\}_n$ converges. Now, by Abel's summation by parts formula, if $S_{\mp k}^r = \sum_{i=1}^{k}(a_{\mp i} - a_{\mp i}^r)$, then, for $1 \le m < n$,

$$|\sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r) - \sum_{k=-m}^{m}{}' \frac{1}{k}(a_k - a_k^r)| \le \sum_{m<|k|\le n} \frac{1}{k(k+1)}|S_k^r|$$
$$+ |\frac{1}{n}S_n^r - \frac{1}{m+1}S_m^r| + |\frac{1}{n}S_{-n}^r - \frac{1}{m+1}S_{-m}^r|.$$

By hypothesis, $\|\mathbf{a}^r - \mathbf{a}\|_\alpha \to 0$, therefore, given $\epsilon > 0$, we can pick N large enough such that whenever $n, r > N$, we have $\|\mathbf{a}^r - \mathbf{a}\|_\alpha < 1$ and $\sum_{k=n}^{\infty} \frac{1}{k^{3-\alpha}\log^\alpha k} < \frac{\epsilon}{2}$.

Then $|\frac{1}{n}S_n^r| \le \frac{1}{n}\sum_{i=-n}^{n}{}' |a_i - a_i^r| \le \frac{1}{n}\frac{n^{\alpha-1}}{\log^\alpha n} \le \frac{1}{n^{2-\alpha}\log^\alpha n}$, hence we have $\lim_n \frac{1}{n}S_n^r \to 0$. Therefore, as $m, n \to \infty$, $|\frac{1}{n}S_n^r - \frac{1}{m+1}S_m^r| \to 0$, and similarly, $|\frac{1}{n}S_{-n}^r - \frac{1}{m+1}S_{-m}^r| \to 0$. Hence, for $m, n, r > N$ (by choosing even larger N, if necessary),

$$\sum_{m<|k|\le n} \frac{1}{k(k+1)}|S_k^r| < \frac{\epsilon}{2}, \quad |\frac{1}{n}S_n^r - \frac{1}{m+1}S_m^r| < \frac{\epsilon}{4} \quad \text{and} \quad |\frac{1}{n}S_{-n}^r - \frac{1}{m+1}S_{-m}^r| < \frac{\epsilon}{4}.$$

Then, it follows that, for $m, n > N$,

$$|\sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r) - \sum_{k=-m}^{m}{}' \frac{1}{k}(a_k - a_k^r)| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon,$$

and hence $\{\sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r)\}_n$ is Cauchy, and hence, converges.

Since

$$\sum_{k=-n}^{n}{}' \frac{1}{k} a_k b_k = \sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r) b_k + \sum_{k=-n}^{n}{}' \frac{1}{k} a_k^r b_k,$$

to prove (b) it is enough to show that $|\sum_{k=-n}^{n}{}' \frac{1}{k}(a_k - a_k^r)b_k - \sum_{k=-m}^{m}{}' \frac{1}{k}(a_k - a_k^r)b_k| \to 0$ as $m, n \to \infty$. Now, letting $n > m$, by the inequality

$$|\sum_{m<|k|\leq n} \frac{1}{k}(a_k - a_k^r)b_k| \leq \|\mathbf{b}\|_\infty \sum_{m<|k|\leq n} \frac{1}{k}|a_k - a_k^r|,$$

the same method used in part (a) proves the assertion. ∎

Remark. Since for any $\lambda \in \mathbb{C},\ |\lambda| = 1$, the sequence $\{\lambda^k\}$ is a Hilbert sequence, for any $\mathbf{a} \in M_\alpha$ and for any $|\lambda| = 1$ the sequence $\{\lambda^k a_k\}$ belongs to M_α, and is a Hilbert sequence.

As an application of Proposition 3.1 we will extend the assertion of Theorem 2.3 to T-admissible processes. Let (X, Σ, μ, T) be a measure preserving system. A family of functions $F = \{f_i\}_{i\in\mathbb{Z}} \subset L_p(X),\ 1 \leq p \leq \infty$, is called a *$T$-admissible process on $\mathbb{Z}$* if $T^{\pm 1} f_{\pm i} \leq f_{\pm(i+1)}$ for $i \geq 0$. When the equality holds, F is called a *T-additive process* and is necessarily of the form $F = \{T^i f\}_{i\in\mathbb{Z}}$, for some $f \in L_p(X)$. A process $F = \{f_i\} \subset L_p$ is called *strongly bounded* when $\sup_{n\in\mathbb{Z}} \|f_n\|_p < \infty$; and it is called *symmetric* if $T^{2i} f_{-i} = f_i$ for all $i \in \mathbb{Z}$.

Given a process $F = \{f_i\}$, define the Hilbert transform of F by $\lim_n H_n F(x)$, where $H_n F(x) = \sum_{i=-n}^{n}{}' \frac{1}{i} f_i(x)$. The eHt of a symmetric strongly bounded T-admissible process F exists a.e. for all $F \subset L_1$ [6]. There, it is also shown that if $F = \{f_n\} \subset L_p$ is a positive symmetric strongly bounded T-admissible process, then there exists a monotone increasing sequence $\{v_r\} \in L_p^+$ and $v_r \uparrow \delta \in L_p$ such that $f_n = T^n v_{|n|}$ for all $n \in \mathbb{Z}$, $f_n \leq T^n \delta$ for all $n \in \mathbb{Z}$, and $\|\delta\|_p = \sup_{n\in\mathbb{Z}} \|f_n\|_p$.

For $r \geq 1$, define $g_i^r(x) = f_i(x)$ for $0 \leq |i| \leq r$ and

$$g_i^r(x) \ = \begin{cases} T^{i-r} f_r(x) & \text{for} \quad i > r \\ T^{-i+r} f_{-r}(x) & \text{for} \quad -i > r. \end{cases}$$

Thus, $g_i^r(x) \leq f_i(x)$ for every $i \in \mathbb{Z}$ and for each $r \geq 1$, and,

$$0 \leq f_i(x) - g_i^r(x) \leq T^i(\delta - v_r)(x) \ \text{ if } |i| > r, \quad \text{and } 0 \text{ if } |i| \leq r.$$

Observe that, $\|\delta - v_r\|_p \downarrow 0$ as $r \to \infty$. Furthermore, ignoring the first r terms, the process $\{g_k^r\}_k$ is T-additive. It follows that, if $\mathbf{a} \in \mathbb{C} \oplus A_\alpha$, then, for each $r \geq 1$, $\lim_n \sum_{-n}^{n}{}' \frac{a_i g_i^r}{i}$ exists a.e. by Theorem 2.3. Therefore, for each $r \geq 1$, for a.e. $x \in X$, the sequence $\{a_i g_i^r(x)\}$ is a Hilbert sequence.

Theorem 3.2 *Let $F \subset L_2$ be a symmetric, strongly bounded T-admissible process. If $\mathbf{a} \in M_\alpha$, then*

$$\lim_n \sum_{-n}^{n}{}' \frac{a_i f_i(x)}{i} \quad \textit{exists a.e.}$$

Proof. Let $\mathbf{u}^r$ and $\mathbf{u}$ be defined by $\mathbf{u}^r = \{a_i g_i^r\}$ and $\mathbf{u} = \{a_i f_i\},\ r \geq 1$, respectively. By the assumptions, for each r, the sequence $\mathbf{u}^r$ is a Hilbert sequence a.e. $x \in X$. Since,

$$0 \leq \frac{\log^\alpha n}{n^{\alpha-1}} \sum_{i=-n}^{n} |a_i g_i^r - a_i f_i| \leq \frac{\log^\alpha n}{n^{\alpha-1}} \sum_{i=-n}^{n} |a_i| T^i(\delta - v_r),$$

it follows that,

$$0 \leq \int \left[\frac{\log^\alpha n}{n^{\alpha-1}} \sum_{i=-n}^{n} |a_i| T^i(\delta - v_r) \right] d\mu \leq \frac{\log^\alpha n}{n^{\alpha-1}} \sum_{i=-n}^{n} |a_i| \|\delta - v_r\|_2 \leq C_{\mathbf{a}} \|\delta - v_r\|_2 \to 0.$$

Hence, by Proposition 3.1 (a), it follows that $\mathbf{u} = \{a_i f_i(x)\}$ is Hilbert sequence for a.e., which proves that $\lim_n \sum_{-n}^{n}{}' \frac{a_i f_i(x)}{i}$ exists a.e. ∎

Corollary 3.3 *Let (X, Σ, μ, T) be a measure preserving system and $\mathbf{a} \in MB_\alpha$ be a two-sided sequence. Then*

$$\lim_n \sum_{-n}^{n}{}' \frac{a_i f_i(x)}{i} \quad \text{exists a.e.}$$

for any symmetric, strongly bounded T-admissible process $F = \{f_k\} \subset L_2(X)$. If $\mathbf{a} \in MB_\alpha$ is a one-sided sequence, then the assertion holds if (X, Σ, μ, T) is a weakly mixing system.

REFERENCES

[1] Assani I. A Wiener-Wintner property for the helical transform, Ergod. Th. & Dynam. Syst., **12**, 185-194, 1992.

[2] Bellow A, Losert V. The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences. Trans. AMS, **288**, 307-345, 1985.

[3] Brunel A, Keane M. Ergodic theorems for operator sequences. Zeit. Wahr., **12**, 231-240, 1969.

[4] Cotlar M. A unified theory of of Hilbert transforms and ergodic theorem. Rev. mat. Cuyana, **1**, 105-167, 1955.

[5] Campbell J, Petersen K. The spectral measure and Hilbert transform of a measure-preserving transformation. Trans. AMS, **313**, 121-129, 1989.

[6] Çömez D. Existence of discrete ergodic singular transforms for admissible processes. Colloq. Math., **112**, 335-343, 2008.

[7] Çömez D. The modulated ergodic Hilbert transform. Discrete & Cont. Dynam. Syst., Series S, **2**, 325-336, 2009.

[8] Jajte R. On the existence of the ergodic Hilbert transform. The Annals of Prob., **15**, 831-835, 1987

[9] Kahane J.P. Sur les coefficients de Fourier-Bohr. Studia Math., **21**, 103-106, 1961.

[10] Lacey M, Terwilleger E. A Wiener-Wintner theorem for the Hilbert transform. Ark. Math., **46**, 315-336, 2008.

[11] Petersen K. Another proof of the existence of the ergodic Hilbert transform. Proc. AMS, **88**, 39-43, 1983.

[12] Sato R. A remark on the ergodic Hilbert transform. Math. J. Okayama Univ., **28**, 159-163, 1986.

[13] Zygmund A. Trigonometric Series. Cambridge University Press 1959.

Azer Akhmedov

Department of Mathematics, North Dakota State University, Department # 2750, PO Box 6050, Fargo, ND 58108-6050, USA.

E-mail address: azer.akhmedov@ndsu.edu

Doğan Çömez

Department of Mathematics, North Dakota State University, Department # 2750, PO Box 6050, Fargo, ND 58108-6050, USA.

E-mail address: Dogan.Comez@ndsu.edu