Amenable subgroups of Homeo($\mathbb{R}$) with large characterizing quotients

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ABSTRACT: We construct a finitely generated solvable subgroup of Homeo$_+(\mathbb{R})$ with non-metaabelian characterizing quotient.

In [Be], the author claims a certain classification result for subgroups of Homeo$_+(\mathbb{R})$ - the group of orientation preserving homeomorphisms of the line. Shortly after the appearance of the first version of his paper, a well known immediate counterexample was pointed out (by us, Matthew Brin, and Andrés Navas). The author has stated he can correct the paper by essentially adding a hypothesis about the existence of a freely acting element.

In this paper we disprove a major claim of the last version of [Be] (Theorem B*). For the sake of completeness let us quote the statement of this theorem from [Be]:

Theorem B*. Let G be a subgroup of Homeo$_+(\mathbb{R})$ with a freely acting element. Then either the quotient group G/H_G is not amenable or the quotient group is solvable with solvability length not greater than 2. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.

We prove the following theorem to contradict this statement.

Theorem 1. There exists a finitely generated solvable subgroup Γ of Homeo$_+(\mathbb{R})$ with a freely acting element such that Γ/H_Γ has solvability length greater than 2.

The subgroup H_Γ is defined in [Be]. For the sake of completeness we will recall the definition of it below but the only thing the reader of this paper needs to know (about the definition of H_Γ) is that a freely acting element of Γ does not belong to H_Γ. The quotient Γ/H_Γ turns out to be a very meaningful object. It has some characterizing power, therefore Theorem B* seemed very interesting to us.

Besides the quotient Γ/H_Γ, another major characteristics is the notion of *minimal set*. Given a subgroup $\Gamma \leq$ Homeo$_+(\mathbb{R})$, a non-empty closed Γ-invariant subset $E \subseteq \mathbb{R}$ is called a minimal set of Γ if it does not contain a proper non-empty closed Γ-invariant subset. If there is

no such set E then by definition we assume that the minimal set is empty.

For finitely generated subgroups of $Homeo_+(\mathbb{R})$, there exists a non-empty minimal set. (cf.[Be] or [N]).

Let us now *quote* the following definition from [Be].

Definition 1. For a subgroup Γ of $Homeo_+(\mathbb{R})$, the normal subgroup H_Γ is defined as follows:

1) if the minimal set (denoted by $E(\Gamma)$) is neither empty nor discrete then

$$H_\Gamma = \{h \in \Gamma \mid E(\Gamma) \subseteq Fix(h)\}$$

2) if the minimal set is non-empty and discrete then $H_\Gamma = \Gamma^s$ (here $\Gamma^s = \cup_{t \in \mathbb{R}} St_\Gamma(t)$, i.e. Γ^s denotes the union of stabilizers of all points $t \in \mathbb{R}$).

3) if the minimal set is empty then $H_\Gamma = 1$.

The reader is referred to [Be] for well definedness of the subgroup H_Γ. Notice that the set Γ^s is not necessarily a subgroup of Γ, in general. However, it is a very nice lemma [Be] that a subgroup generated by Γ^s either coincides with Γ^s or coincides with Γ itself.

Remark 1. The condition about existence of a freely acting element is indeed very interesting. For example, if all non-identity elements of a subgroup Γ of $Homeo_+(\mathbb{R})$ act freely then the group is Archimedean with a bi-invariant order, and therefore (by Hölder's Theorem) it is Abelian ([cf.N]). If every non-identity element has at most one fixed point then the group is meataabelian, even more specifically, it is isomorphic to a subgroup of the affine group $Aff(\mathbb{R})$ as proved by Barbot [Ba] and Kovacevic [K] (see [FF] for the history of this result). If every non-identity element has at most N fixed points, where N is a fixed positive integer, then we do not know what are the algebraic implications of this condition but it seems to us that this is an enormous restriction on the group. For example, if Γ contains two distinct elements a, b such that $a^m = b^m$ for some non-zero integer m (for example, the Klein bottle group $K = \langle a, b \mid a^2 = b^2 \rangle$), then such Γ cannot satisfy the above condition for any fixed N - the element ab^{-1} necessarily has infinitely many fixed points.

Note: We are thankful to N.Guelman, A.Navas, and C.Rivas for their interest in this work. In the first version of this paper, a long overlooked error was pointed out to me by C.Rivas and N.Guelman.

PROOF OF THEOREM 1.

We intend to construct a finitely generated solvable subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ such that Γ contains a freely acting element and Γ/H_Γ is not metaabelian. The only thing we need to know about the definition H_Γ is that a freely acting element does not belong to it.

Let Γ be a group generated by two elements $t, a \in \Gamma$. Let us assume that the following conditions hold:

(i) Γ is a finitely generated solvable group of derived length at least three;

(ii) Γ is left-orderable with a left order $<$;

(iii) There exists $g \in \Gamma^{(2)}$ such that for all $f \in \Gamma$, there exists a positive integer n such that $g^{-n} < f < g^n$.

We are postponing the construction of Γ with properties (i)-(iii) till the end.

Because of (i)-(iii), Γ is embeddable in $\mathrm{Homeo}_+(\mathbb{R})$. Moreover, we can embed Γ faithfully in $\mathrm{Homeo}_+(\mathbb{R})$ such that the following conditions hold:

(c1) if $g_1, g_2 \in \Gamma, g_1 < g_2$ then $g_1(0) < g_2(0)$ (in particular, $g(0) > 0$ for all positive $g \in \Gamma$);

(c2) Γ has no fixed point.

Let us also observe that if all conditions (i)-(iii) and (c1)-(c2) hold then the element g of $\Gamma^{(2)}$ necessarily acts freely.

Construction of Γ: Let us now construct Γ with properties (i)-(iii). We will write $\mathrm{Sol}(2, d)$ to denote the free solvable group of derived length $d \geq 1$ on the set $\mathcal{A} = \{a, b\}$. The group Γ will be isomorphic to a certain subgroup of $\mathrm{Sol}(2, 3)$; so Γ has a relatively simple algebraic structure. However, we will put a left order in it which is not induced by the most natural left order that one considers in $\mathrm{Sol}(2, 3)$.

Let also F, L be the free group and the free metaabelian group on the same alphabet $\mathcal{A}$ respectively, and $\mathcal{G}(L, \mathcal{A})$ be the Cayley graph of L with respect to the generating set $\mathcal{A}$. Any element $w \in \mathrm{Sol}(2, 3)$ can be written as a word $W(a, b) \in F$ in the alphabet $\mathcal{A}$. This word is not unique but if W_1 is another such word, then W and W_1 go over every oriented edge e of $\mathcal{G}(L, \mathcal{A})$ the same number of times. We will write $N(e, w)$ to denote this number. For a word $w \in F$, we will write $\overline{w}$ to denote the same word in L.

For every oriented edge $e = (u, v)$ in the Cayley graph $\mathcal{G}(L, \mathcal{A})$, we have either $v = u\omega$ or $v = u\omega^{-1}$ where $\omega \in \{a, b\}$; in the former case we call the edge e *positive*, and in the latter case we call it *negative*. The vertices u, v will be called *the start* and *the end* of the edge e respectively; we will also write $u = start(e)$ and $v = end(e)$.

An element $w \in L$ will be called *an elementary loop* if, as a word in the alphabet $\mathcal{A}$, it can be written as $W(a, b) = g_1 aba^{-1}b^{-1}g_1^{-1}$ or $W(a, b) = g_2 bab^{-1}a^{-1}g_2^{-1}$ where g_1 (g_2) is represented by a path $a^m b^n$ in the Cayley graph $\mathcal{G}(L, \mathcal{A})$. In the former case, w will be called *positive* and in the latter case it will be called *negative*. The point $(n, m) \in \mathbb{Z}^2$ will be called the *neck of the loop* w. Now, an element $w \in \mathrm{Sol}(2, 3)$ will be called a positive (negative) loop if it is represented by a word $W = W(a, b)$ such that the word $\overline{W}$ is a positive (negative) loop.

Let $S = \{s_1, s_2, \dots\}$ be a set of positive loops of $\mathrm{Sol}(2, 3)$ such that for all $m, n \in \mathbb{Z}$, there exists $i \geq 1$ such that $s_i = a^m b^n [a, b] b^{-n} a^{-m}$, moreover, s_i and s_j have different necks for all $i \neq j$. Let also G be the subgroup of $\mathrm{Sol}(2, 3)$ generated by S, and L' be the free Abelian group over the set S. Again, by an abuse of notation, for a word w representing an element of G, we will write $\overline{w}$ for the same word in L'. We let $\mathcal{G}(G, S)$ be the Cayley graph of G with respect to the generating set S, and $\mathcal{G}(L', S)$ be the Cayley graph of L' with respect to the generating set S. In the Cayley graph $\mathcal{G}(L', S)$ we define positive and negative edges similarly: for an edge $e = (u, v)$ in the Cayley graph $\mathcal{G}(L', S)$, we have either $v = u\omega$ or $v = u\omega^{-1}$ where $\omega \in S$; in the former case we call the edge e *positive*, and in the latter case we call it *negative*.

Let also $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \dots$ be positive real numbers which are algebraically independent over the rationals (i.e. there exists no non-zero polynomial $P(z_1, \dots, z_n)$ over $\mathbb{Z}$ such that $P(a_1, \dots, a_n) = 0$, for some n distinct elements $a_1, \dots, a_n$ of the set $\{\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \dots\}$), and $\delta_1, \delta_2, \beta_1, \beta_2, \beta_3, \dots$ be positive real numbers rationally independent over the field generated by $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3, \dots$.

For every $w \in \mathrm{Sol}(2, 3)$ we can write w as a a reduced word $w = W(a, b) = \underline{x_1 x_2 \dots x_n}$ where $x_i \in \{a, a^{-1}, b, b^{-1}\}, 1 \leq i \leq n$. Then the word $\overline{w} = \overline{W(a, b)}$ represents an element $a^m b^n \in L$ for some $m, n \in \mathbb{Z}$, and we can define $\lambda_1(w) = \gamma_1^m \gamma_2^n$, and

$$\Lambda_1(w) = \sum_{e \in E_+(W)} \beta(e) N(e, w) \lambda_1(start(e)) - \sum_{e \in E_-(W)} \beta(e) N(e, w) \lambda_1(end(e))$$

where $E_+(W), E_-(W)$ denote the set of positive and negative edges of the path $\overline{W}$ respectively; $\beta(e) = \delta_1$ if e is parallel to the edge $(1, a)$, and $\beta(e) = \delta_2$ otherwise.

The quantity $\Lambda_1(w)$ does not depend on the choice of the representative word W. Moreover, for all $w, w_1, w_2 \in \mathrm{Sol}(2, 3)$, if $\Lambda_1(w_1) > \Lambda_1(w_2)$ then $\Lambda_1(ww_1) > \Lambda_2(ww_2)$. Indeed, we will have $\Lambda_1(ww_1) = \Lambda_1(w) + \lambda_1(w)\Lambda_1(w_1)$; similarly, $\Lambda_1(ww_2) = \Lambda_1(w) + \lambda_1(w)\Lambda_1(w_2)$.

However, if w belongs to the second commutator subgroup of $\mathrm{Sol}(2, 3)$ then $\Lambda_1(w) = 0$. On the other hand, if w does not belong to the second commutator subgroup of $\mathrm{Sol}(2, 3)$ then by the choice of $\gamma_1, \gamma_2, \delta_1, \delta_2$ we have $\Lambda_1(w) \neq 0$. This allows us to define a left order in $\mathrm{Sol}(2, 2)$ (instead of $\mathrm{Sol}(2, 3)$) as follows: for $g_1, g_2 \in G$ we let $g_1 < g_2$ if $\Lambda_1(g_1) < \Lambda_1(g_2)$.

Now we will find a similar left order in a certain subgroup $\mathrm{Sol}(2, 3)$ which has derived length three. For every $w \in G$ we can write w as a reduced word $W(s_1, s_2, \dots) = x_1 x_2 \dots x_n \in G$ where $x_i \in S \cup S^{-1}, 1 \le i \le n$. Then $\overline{w} = \prod_{i \ge 1} \overline{s}_i^{n_i}$ for some integer exponents $n_1, n_2, \dots$. We define $\lambda(w) = \prod_{i \ge 1} \alpha_i^{n_i}$ and

$$\Lambda(w) = \sum_{e \in E_+(W)} \beta(e) N(e, w) \lambda(start(e)) - \sum_{e \in E_-(W)} \beta(e) N(e, w) \lambda(end(e))$$

where $W \in L'$ is a representative of the element $w \in G$, and $E_+(W), E_-(W)$ denote the set of positive and negative edges of the path $\overline{W}$ respectively, in the Cayley graph $\mathcal{G}(L', S)$; also, if e is parallel to the edge $(1, s_i)$ then we let $\beta(e) = \beta_i$.

Let us emphasize that the quantity $\Lambda(w)$ does not depend on the choice of the representative word W. Moreover, for all $w, w_1, w_2 \in G$, if $\Lambda(w_1) > \Lambda(w_2)$ then $\Lambda(ww_1) > \Lambda(ww_2)$. This allows us to define the left order in G as follows: for $g_1, g_2 \in G$ we let $g_1 < g_2$ if $\Lambda(g_1) < \Lambda(g_2)$. Thus G satisfies conditions (i) and (ii). However, G is metaabelian since it is a subgroup of the commutator subgroup of $\mathrm{Sol}(2, 3)$.

Now we are ready to introduce the group Γ. To do this, we take the loops

$$x = a[a, b]a^{-1}, y = a^2[a, b]a^{-2}, z = a^4[a, b]a^{-4}, t = a^8[a, b]a^{-8}$$

and let Γ be the subgroup of $\mathrm{Sol}(2, 3)$ generated by the elements

$$\theta = bx, \eta = b^2 y, \xi = b^4 z, \zeta = b^8 t.$$

Any element $w \in \Gamma$ can be written canonically as $w = b^n u$ or $w = u b^n$ where $u \in G, n \in \mathbb{Z}$. We extend the map $\Lambda : G \to \mathbb{R}$ to the subgroup Γ as follows: if $w = b^n u$ we let $\Lambda(w) = \Lambda(b^n u b^{-n})$, and if $w = u b^n$ we let $\Lambda(w) = \Lambda(u)$.

Now, we make an important observation that if $w \in \Gamma^{(2)}$ then $\Lambda(w^m) = m\Lambda(w)$.

Now, we let $g = [[\theta, \eta], [\xi, \zeta]]$. By replacing g with g^{-1} if necessary it is immediate to see that g satisfies condition (iii). Indeed, without loss of generality we may denote the positive loops with the neck at the points $(1, 1), (3, 2), (3, 1), (2, 2), (4, 4), (12, 8), (12, 4), (8, 8)$ with $s_1, \ldots, s_8$ respectively. Then

$$\Lambda(g) = \beta_1 + \alpha_1\beta_2 - \alpha_1\alpha_2\alpha_3^{-1}\beta_3 - A\beta_4 + A\beta_5 + A\alpha_5\beta_6 - A\alpha_5\alpha_6\alpha_7^{-1}\beta_7 - AB\beta_8 +$$
$$+ AB\beta_4 + \alpha_1\alpha_2\alpha_3^{-1}B\beta_3 - \alpha_1 B\beta_2 - B\beta_1 + B\beta_8 + \alpha_5\alpha_6\alpha_7^{-1}\beta_7 - \alpha_5\beta_6 - \beta_5$$

where $A = \alpha_1\alpha_2\alpha_3^{-1}\alpha_4^{-1}$ and $B = \alpha_5\alpha_6\alpha_7^{-1}\alpha_8^{-1}$.

Finally, it remains to notice that, by the choice of the coefficients $\alpha_j, \beta_j, j \geq 1$ we have $\Lambda(g) \neq 0$.

Remark 2. The construction for the proof of Theorem 1 can be generalized to obtain a finitely generated solvable group Γ of an arbitrary derived length n such that $\Gamma^{(n)}$ possesses freely acting elements.

R e f e r e n c e s

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