FINITENESS OF THE TOPOLOGICAL RANK OF DIFFEOMORPHISM GROUPS

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Abstract. For a compact smooth manifold M (with boundary) we prove that the topological rank of the diffeomorphism group $\mathrm{Diff}_0^k(M)$ ($\mathrm{Diff}_0^k(M, \partial M)$) is finite for all $k \geq 1$. This extends a result from [2] where the same claim is proved in the special case of $dimM = k = 1$.

1. Introduction

The *topological rank* of a separable topological group G, denoted by $\mathrm{rank}_{top}(G)$, is the minimal integer $n \geq 1$ such that for some n-tuple $(g_1, \dots, g_n)$ of elements of G the group $\Gamma = \langle g_1, g_2, ..., g_n \rangle$ which they generate is dense in G. In this case we say G is *topologically n-generated* and the n-tuple $(g_1, \dots, g_n)$ *topologically generates* G.

It is not difficult to see that the topological tank of $\mathbb{R}^n$ equals $n+1$ while the topological rank of the n-torus $\mathbb{T}^n$ equals 1. It is also immediate to see that a 1-generated group is necessarily Abelian.

Since $\mathrm{rank}_{top}(\mathbb{T}^n) = 1$, all compact connected solvable Lie groups have topological rank equal to 1. On the other hand, it has been proved by Kuranishi [11] that all compact semisimple Lie groups are 2-generated. Hofmann and Morris [9] have extended this result to all separable compact connected groups. Very recently, applying the solution of Hilbert's Fifth Problem, Gelander and Le Maitre [4] have shown that every separable connected locally compact group is topologically finitely generated.

In the realm of more general Polish groups, interesting finiteness results have been obtained by Macpherson in [16], and by A. Kechris and C. Rosendal in [13], for the topological ranks of the following groups: S_∞ (the group of bijections of $\mathbb{N}$), the automorphism group of the (so-called) random graph (see [16]), $H(2^\mathbb{N})$ (the homeomorphism group of the Cantor space), $H(2^\mathbb{N}, \sigma)$ (the group of the measure preserving homeomorphisms of $2^\mathbb{N}$ with the usual product measure), $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ (the automorphism group of the infinitely splitting rooted tree), and $\mathrm{Aut}([0,1], \lambda)$ (Lebesgue measure preserving automorphisms of the closed interval), see [13] (as noted in [13], that the latter group is topologically 2-generated is first shown in [6] and [17] by different means). In all these examples, it is shown that the groups in question admit a *cyclically dense conjugacy class* [1] thus the topological rank equals two. In particular, all these groups have a *Rokhlin property*, i.e. they possess a dense conjugacy class.

In [2], the finiteness of the ranks of the diffeomorphisms groups of compact 1-manifolds has been proved. These diffeomorphism groups are quite different in many regards from the groups mentioned in the previous paragraph, in fact, one can almost view them as Polish groups at the other end of the spectrum. The current paper can be viewed as a continuation of [2]; here, we establish finiteness results for the topological rank of the diffeomorphism group $\mathrm{Diff}_0^k(M)$ (the connected component of the group of all diffeomorphisms of M) for an arbitrary compact smooth manifold M, as well as for the groups $\mathrm{Diff}_0^k(M, \partial M)$ for an arbitrary compact smooth manifold M with boundary. Here, $\mathrm{Diff}_0^k(M, \partial M)$ denotes the connected component of the group of all diffeomorphisms of M fixing the boundary ∂M pointwise. We are also motivated to understand discrete and dense finitely generated subgroups of these groups, seeking parallels with the very rich and mostly established theory of discrete and dense subgroups of Lie groups. The reader may consult the paper [1] for a large number of remarks and open questions in this program where mainly the group $\mathrm{Diff}_+(I)$ has been considered, however, quite many of the questions there are still meaningful for a general compact smooth manifold and in the higher regularity as well.

We would like to emphasize that the easier case of a finiteness of the rank of $\mathrm{Homeo}_0(M)$ for compact topological manifolds (with boundary) also follows from the proof, although this is a much weaker result. In fact, it is quite well known that the group $\mathrm{Homeo}_+(I)$ of orientation-preserving homeomorphisms of the interval $I = [0, 1]$ is topologically 2-generated, as the Thompson's group F embeds there densely. The standard faithful representation of F can be modified to obtain C^∞ smooth representation [5], however, this smoothed out representations is not dense in the C^1-metric, quite the opposite, it is in fact C^1-discrete.

The homeomorphism groups $\mathrm{Homeo}_0(M)$ resemble compact spaces in many ways (especially for the cases of interval I, the disc D^2 and the 2-sphere $\mathbb{S}^2$), while already for the the case of the interval $M = I$,

[1] a group G is said to admit a cyclically dense conjugacy class if for some $f, g \in G$, the set $\{f^n g f^{-n} : n \in \mathbb{Z}\}$ is dense in G.

and in the regularity $k = 1$, the group $\mathrm{Diff}_+(I)$ is homeomorphic to an infinite dimensional Banach space. [2]

The following theorem is proved in [2] in the case of $k = 1$.

Theorem 1.1. *For each $1 \leq k \leq \infty$, there exists a finitely generated dense subgroup of $\mathrm{Diff}^k_+(I)$.*

Our construction relies on the perfectness results of Tsuboi [18] for diffeomorphism groups of the interval, as well as the following lemma on approximation by diffeomorphisms with iterative n-th roots which is independently interesting.

Lemma 1.2. *Let $f \in \mathrm{Diff}^k_+(I)$ without a fixed point in (0,1) and assume that f is C^k-tangent to the identity at 0 and 1. Then for every $r > 0$, there exists a $g \in \mathrm{Diff}^k_+(I)$ and a positive integer N such that g is r-close to identity and g^N is r-close to f, in the C^k metric.*

Lemma 1.2 is an extension of the Lemma 1.2 from [2] to the higher regularity. The proof of this lemma somewhat follows the one of Lemma 1.2 from [2] but it turns out to be much more involved. As for the perfectness result, we use a more subtle version of it from [18].

By the standard fragmentation argument the following corollary is immediate.

Corollary 1.3. For each $1 \leq k \leq \infty$, there exists a finitely generated dense subgroup of $\mathrm{Diff}^k_+(\mathbb{S}^1)$.

The main construction in the proof of Theorem 1.1 generalizes to the case of an arbitrary (smooth) manifold. We then use an analogue of Tsuboi's perfectness result for the diffeomorphism groups of arbitrary compact (smooth) manifolds, recently re-proven by S.Haller, T.Rybicki and J.Teichmann, [10]. Then we find a suitable extension of the Lemma 1.2 in the general manifold case thus obtaining the following more general

Theorem 1.4. *Let M be a compact smooth manifold (with boundary). Then for all $k \geq 1$ the group $\mathrm{Diff}^k_0(M)$ ($\mathrm{Diff}^k_0(M, \partial M)$) admits a finitely generated dense subgroup.*

It should be noted that higher dimensional perfectness results for diffeomorphism groups appear already in the works of M.R.Herman [7, 8] and W.Thurston [19]. The perfectness of the diffeomorphism group

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[2] Any two infinite dimensional separable Banach spaces are indeed homeomorphic by a result of M.I.Kadets [12]; one can also establish an explicit homeomorphism by the map $f \to \ln f'(t) - \ln f'(0)$ from $\mathrm{Diff}^1_+(I)$ to the Wiener space $C_0[0,1]$ - the space of continuous functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$.

$\mathrm{Diff}_c^\infty(M)_0$ for a smooth manifold M has been proved by D.Epstein in [3] which in turn uses the ideas of J.Mather in [14], [15] who dealt with the C^k case, $1 \leq k < \infty,\ k \neq n+1$. In these works, for an arbitrary diffeomorphism $g \in \mathrm{Diff}_c^\infty(M)$, a representation $g = [u_1, v_1] \dots [u_N, v_N]$ is guaranteed but seems with less control on u_i, v_i, and with a weaker bound on N. In [10], besides establishing the *uniform perfectness* with a very strong bound, also the so-called *smooth perfectness* of the group $\mathrm{Diff}_c^\infty(M)_0$ has been proved.

2. Finiteness of the rank in higher regularity and in higher dimensions

In this section, we will prove Theorem 1.1 and Theorem 1.4 modulo the axillary results Lemma 1.2 and the extension of it in the n-dimensional case, Lemma 3.9. The general scheme of the major construction follows the one of [2].

We would like to make an important observation that the proof of Theorem 2.1 in [2] works in showing that $\mathrm{Diff}_+^k(I)$ is finitely generated for all $k \geq 1$. Indeed, in the proof of Theorem 2.1, one only needs to make the following modifications:

First, the group G is redefined to be equal to

$$\{f \in \mathrm{Diff}_+^k(I) \mid f \text{ is } C^k\text{−flat}\}$$

where C^k-*flatness* means that $f'(0) = f'(1) = 1$ and $f^{(j)}(0) = g^{(j)}(0) = 0, 2 \leq j \leq k$. Secondly, the C^1-metric everywhere is replaced with C^k-metric, and the C^1-spaces are replaced with C^k-spaces.

Thirdly, the condition (b-i) is replaced with
(b-i)′ $\theta'(a_1) = \theta'(b_1) = 1$ and $\theta^{(j)}(a_1) = \theta^{(j)}(b_1) = 0$ for all $2 \leq j \leq k$ where θ is any of the functions f, g, u and v.
Also, the condition (c-i) is replaced with the following condition:
(c-i)′ for all $x \in \{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$, $f'(x) = g'(x) = u'(x) = v'(x) = 1$ and $f^{(j)}(x) = g^{(j)}(x) = u^{(j)}(x) = u^{(j)}(x) = 0, 2 \leq j \leq k$;

and the condition (d-i) is replaced with
(d-i)′ $f'(x) = g'(x) = u'(x) = v'(x) = 1$ and $f^{(j)}(x) = g^{(j)}(x) = u^{(j)}(x) = v^{(j)}(x) = 0, 2 \leq j \leq k$, for all $x \in \{a_{n+1}, b_{n+1}\}$.

Besides, we also assume that the function h is C^k-flat (instead of being C^1-flat) at the endpoints of the intervals $J_n, n \geq 1$.

In the main body of the construction, instead of the simple version of the Tsuboi's perfectness result (Lemma 3.3 in [2]) we use Lemma 3.6 of Tsuboi which is a more subtle perfectness result in the higher regularity. Finally, in the conclusion, we use Lemma 3.5 instead of Lemma 3.2.

Now, we would like to extend the result to all groups $\mathrm{Diff}_0^k(M)$ ($\mathrm{Diff}_0^k(M, \partial M)$) where M is a compact smooth manifold (with boundary) with C^k topology.

Theorem 2.1. *If M is a compact smooth manifold then the group $\mathrm{Diff}_0^k(M)$ is topologically finitely generated. Similarly, if M is a compact smooth manifold with boundary then the topological rank of $\mathrm{Diff}_0^k(M, \partial M)$ is finite.*

Proof. We may and will assume that M is connected.

First, we will consider *the case of a closed manifold.* Let $d = \mathrm{dim}M$, $\Omega_1, \dots, \Omega_m$ be open charts of M covering the entire manifold M (the open sets $\Omega_s, 1 \leq s \leq m$ are diffeomorphic to a unit open ball in $\mathbb{R}^d$). In each $\Omega_s, 1 \leq s \leq m$ we take a sequence of open sets $D_1^{(s)}, D_2^{(s)}, D_3^{(s)}, \dots$ accumulating to some point $p_s \in \Omega_s \backslash (\underset{n\geq 1}{\cup} \overline{D_n^{(s)}})$ such that the closures $\overline{D_1^{(s)}}, \overline{D_2^{(s)}}, \overline{D_3^{(s)}}, \dots$ are mutually disjoint. Since Ω_s can be identified with an open unit ball, we can and will take the open sets $D_1^{(s)}, D_2^{(s)}, \dots$ to be open balls in Ω_s with centers lying on a segment and accumulating to p from one side. Let $\phi_s, \psi_s, 1 \leq s \leq m$ be diffeomorphisms of M such that the conditions analogous to conditions (a-i)-(a-iii) of [2] hold, more precisely,

(i) $Int(\mathrm{supp}(\psi_s)) = \Omega_s$,

(ii) $\phi_s^{-1}(D_n^{(s)}) = D_{n+1}^{(s)}, \forall n \geq 1$,

(iii) the sets $A_k = \psi^{-k} D_1^{(s)}$ form an increasing chain of open sets in $D_1^{(s)}$ (i.e. $A_1 \subseteq A_2 \subseteq \dots \subseteq D_1^{(s)}$) and $\underset{k\geq 1}{\cup} A_k = \Omega_s$.

Notice that the group $\mathrm{Diff}_0^k(M)$ is generated by subgroups $G_s = \{\eta \in \mathrm{Diff}_0^k(M) \mid \mathrm{supp}(\eta) \cup \mathrm{Im}(\eta) \subseteq \overline{\Omega_s}\}$. Hence it suffices to build maps $f_s, g_s, u_s, v_s, h_s \in G_s$ such that $f_s, g_s, u_s, v_s, h_s, \phi_s, \psi_s$ generate G_s.

Now, fix $s \in \{1, \dots, m\}$ and let $\eta_1, \eta_2, \dots$ be dense sequence in the subset $\{[\omega_1\omega_2, \omega_3\omega_4] : \omega_i \in G_s, 1 \leq i \leq 4\}$. We will identify the set Ω_s with the open unit ball in $\mathbb{R}^d$.

We define f, g, u, v and h on $M \setminus \bigsqcup_{n\geq 1} D_n^{(s)}$ to be an identity map and build these maps on the open sets $D_1^{(s)}, D_2^{(s)}, \dots$ inductively as in the proof of Theorem 2.1 from [2]. More precisely, let F_0 be a free group formally generated by f, g, u, v, h and $(\epsilon_n)_{n\geq 1}$ be a positive sequence decreasing to zero.

In the domain $D_1^{(s)}$ we define f, g, u, v and h such that all of them map $D_1^{(s)}$ to itself and all are C^k-flat at the boundary $\partial(D_1^{(s)})$; moreover, for some word $X_1(f, g, u, v)$ in F_0 we have $\phi^{-1}\psi^{m_1}\phi X_1(f, g, u, v)\phi^{-1}\psi^{-m_1}\phi$ is ϵ_1 close to η_1 for some sufficiently big m_1.

Suppose now f, g, u, v and h are defined on $\bigsqcup_{1\leq i\leq n} D_i^{(s)}$, as well as the words $X_i(f, g, u, v, h), Y_i(f, g), Z_i(u, v), 1 \leq i \leq n$ are defined such that all the maps f, g, u, v, h are mapping each of $D_i^{(s)}, 1 \leq i \leq n$ to itself, and the following conditions analogous to (b-i)-(b-iv) of [2] hold:

(i) for all $x \in \partial D_i^{(s)}$ and for all $l, j \in \{1, \dots, d\}$,

$$(\partial f_l/\partial x_j)(x) = (\partial g_l/\partial x_j)(x) = (\partial u_l/\partial x_j)(x) = (\partial v_l/\partial x_j)(x) = (\partial h_l/\partial x_j)(x) = \delta_l^j$$

and for all $2 \leq r \leq k, j_1, \dots, j_r, l \in \{1, \dots, d\}$,

$$(\partial^r \theta_l/\partial x_{j_1} \dots \partial x_{j_r})(x) = 0$$

where θ_l is any of the functions f_l, g_l, u_l, v_l, h_l;

(ii) f, g, u, v and h are ϵ_i-close to the identity in $C^k(D_i^{(s)})$;

(iii) for some natural number p_i we have

$$(\mathrm{supp}(h^p Y_j(f, g)h^{-p}) \cap \mathrm{supp}(Z_j(u, v))) \cap D_i^{(s)} = \emptyset$$

for all $1 \leq i \leq n, 1 \leq j < i$ and $p \geq p_i$;

(iv) for the word $X_i = [h^{p_i} Y_i(f, g)h^{p_i}, Z_i(u, v)]$ in F_0 we have

$$\phi^{-i}\psi^{m_i}\phi^i X_i(f, g, u, v, h)\phi^{-i}\psi^{-m_i}\phi^i$$

is ϵ_i-close to η_i for some sufficiently big m_i.

Then we define the diffeomorphisms $f = (f_1, \dots, f_d), g = (g_1, \dots, g_d), u = (u_1, \dots, u_d), v = (v_1, \dots, v_d), h = (h_1, \dots, h_d)$ on the domain $D_{n+1}^{(s)}$ such that all these maps are mapping $D_{n+1}^{(s)}$ to itself, and

(i) for all $x \in \partial D_{n+1}^{(s)}$ and for all $l, j \in \{1, \dots, d\}$,

$$(\partial f_l/\partial x_j)(x) = (\partial g_l/\partial x_j)(x) = (\partial u_l/\partial x_j)(x) = (\partial v_l/\partial x_j)(x) = (\partial h_l/\partial x_j)(x) = \delta_l^j$$

and for all $2 \leq r \leq k, j_1, \dots, j_r, l \in \{1, \dots, d\}$,

$$(\partial^r \theta_l/\partial x_{j_1} \dots \partial x_{j_r})(x) = 0$$

where θ_l is any of the functions f_l, g_l, u_l, v_l, h_l;

(ii) f, g, u, v and h are ϵ_{n+1}-close to the identity in $C^k(D^{(s)}_{n+1})$;

(iii) for some natural number p_{n+1} we have

$$(\mathrm{supp}(h^p Y_i(f,g)h^{-p}) \cap \mathrm{supp}(Z_i(u,v))) \cap D_i^{(s)} = \emptyset$$

for all $1 \leq i \leq n, p \geq p_{n+1}$;

(iv) for some words $Y_{n+1}(f,g), Z_{n+1}(u,v)$ and for the word $X_{n+1} = [h^{p_{n+1}} Y_{n+1}(f,g) h^{p_{n+1}}, Z_{n+1}(u,v)]$ in F_0 we have

$$\phi^{-(n+1)} \psi^{m_{n+1}} \phi^{n+1} X_{n+1}(f,g,u,v,h) \phi^{-(n+1)} \psi^{-m_{n+1}} \phi^{n+1}$$

is ϵ_{n+1}-close to η_{n+1} for some sufficiently big m_{n+1}.

By induction, we extend f, g, u, v and h to the whole domain Ω_s. By construction, the group generated by ϕ, ψ, f, g, u, v and h is dense in G_s. Indeed, we can take the words $Y_n(f,g) = f^{M_1} g^{M_2}$ and $Z_n(u,v) = u^{M_3} v^{M_4}$ for sufficiently big M_1, M_2, M_3, M_4, and the claim now follows from Lemma 3.9 and Lemma 3.10.

For the *case of a compact manifold with boundary,* let us first notice that the subgroup $\mathrm{Diff}^k(M, \partial M)_{flat}$ of C^k-flat diffeomorphisms of $\mathrm{Diff}^k(M, \partial M)$ is also topologically finitely generated by the treatment of the case of closed manifolds. Secondly, if N is a closed manifold then the group

$G_N = \{f \in \mathrm{Diff}^k_0(N \times I) \mid \forall x \in N, f(x,0) = (x,0), \text{ and } \forall x \in N, \forall t \in I, f(x,t) = (y,t), \text{ for some } y \in N\}$ is topologically finitely generated as well similarly to the case of a closed manifold. Now, by Collar Neighborhood Theorem, the boundary ∂M has an open neighborhood in the form $\partial M \times [0,2)$ in M. Denote $N = \partial M$. Then both of the groups G_N and $\mathrm{Diff}^k_c(M, \partial M)_{flat}$ have finite topological ranks. But these two groups generate $\mathrm{Diff}^k_0(M, \partial M)$ thus we obtain that the topological rank of the latter is finite. $\square$

Remark 2.2. It follows from the proof that the group $\mathrm{Diff}^k_+(I)$ is generated by $3k+7$ elements, and the group $\mathrm{Diff}^k_0(M)$ is generated by $(3k+7)m$ elements where m is the minimal number of charts covering M. With more careful analysis it is possible to decrease this bound significantly. Indeed, it is quite easy to make the bound independent of m; in the case of a closed manifold, one simply needs to take a diffeomorphism $\pi : M \to M$ and an open chart Ω_1 such that $(\pi^n(\Omega_1))_{n \geq 0}$ is an open cover of M (similar arrangements can be made for manifolds with boundary as well). It is also possible to make the bound independent of k. We suspect that in the case of a closed manifold the minimal of generators can be made equal to two. It also follows from the construction that the generators can be chosen from an arbitrary small neighborhood of identity (thus the so called *local rank* is also finite), moreover, all the generators can be chosen from the C^{∞} class.

3. Roots of diffeomorphisms and perfectness results in diffeomorphism groups

In this section we will prove the Lemma 1.2, and then generalize this result to a higher dimensional case. We will need the following

Definition 3.1. Let $l \geq 1$. A sequence $(t_i)_{1\leq i\leq N}$ of real numbers is called *l-quasi-monotone* if there exists $1 < i_1 \leq \dots \leq i_s < N$ for some $s \in \{1, \dots, l\}$ such that each of the subsequences $(t_{i_j}, \dots, t_{i_{j+1}})$ is monotone where $0 \leq j \leq s$ and $i_0 = 1, i_{s+1} = N$. A 1-quasi-monotone sequence will be simply called *quasi-monotone*.

In [2], we prove the following lemma (we recommend that the reader studies the proof of it, since in this section, we are making a reference to the proof.)

Lemma 3.2. *Let $f \in \mathrm{Diff}_+(I)$, $f'(0) = f'(1) = 1$ and $r > 0$. Let also f has no fixed point in $(0,1)$. Then there exists a natural number $N \geq 1$ and $g \in \mathrm{Diff}_+(I)$ such that $||g||_1 < r$ and $||g^N - f||_1 < r$.*

Now, we would like to extend the proof of the above lemma to the case when $f \in \mathrm{Diff}_+^k(I)$ hence we would like to find an approximate root g of f from a small neighborhood of identity in the C^k metric. Here, we need estimations at all levels up to the k-th derivative. Thus, instead of inequalities **(1)** and **(2)** from the proof of Lemma 1.2 in [2]), we will need $k+1$ inequalities. As before, we will define the values of g at the selected points $z_i, 0 \leq i \leq nN$, then we will define the values of the derivatives at those points, then the values of the second derivatives and so on. Then, we extend g to the whole interval $[0,1]$ such that $g'(x)$ will be monotone at each subinterval (z_i, z_{i+1}), moreover, the higher derivatives $g^{(j)}(z)$ do not deviate much from the values $g^{(j)}(z_i), g^{(j)}(z_{i+1})$ at the end points; let us also emphasize that we do not demand the monotonicity of the higher derivatives at the subintervals (z_i, z_{i+1}). In the proof of Lemma 3.2, we also used the l-quasi-monotonicity of the sequence $g'(z_0), \dots, g'(z_{nN})$, i.e. it can divided into at most l (not necessarily strictly) monotone subsequences (in the simplest case of 1-quasi-monotonicity, this means that for all $1 \leq i \leq nN - N$ there exists $p \in \{i, \dots, i+N-1\}$ such that both of the sequences $(g'(z_i), \dots, g'(z_{i+p}))$ and $(g'(z_{i+p}), \dots, g'(z_{i+N}))$ are monotone). This follows directly from the Mean Value Theorem applied to all j-th order derivatives, $0 \leq j \leq k$, of iterates of $f^s, 1 \leq s \leq n$ where n will be chosen in the proof. Also, we will not impose a quasi-monotonicity condition on any of the sequences $g^{(j)}(z_0), \dots, g^{(j)}(z_{nN})$ for any $j \geq 2$.

The following two lemmas are very technical although in essence they are very simple claims. These lemmas can also serve as a guide to the proof of the root approximation lemma, and alert the reader about the strategy of the proof. To state the first lemma, we need to introduce some terminology.

Let $u \in \mathrm{Diff}_+(I)$, n, N be positive integers, and $0 = z_0 < z_1 < \dots < z_{nN-1} < z_{nN} = 1$ such that $z_i = \frac{i}{N^2}, 0 \leq i \leq N$, and $z_{i+1} = u(z_i), N \leq i \leq nN - 1$. In this case we say $(z_i)_{0 \leq i \leq nN}$ is an (n, N)-*associated sequence* of u. For every $j \geq 1, 0 \leq i \leq nN - j$, inductively on j, we define the quantities

$$\overline{u^{(j+1)}(z_i)} = \frac{\overline{u^{(j)}(z_{i+1})} - \overline{u^{(j)}(z_i)}}{z_{i+1} - z_i}$$

where we assume that $\overline{u^{(0)}(z_i)} = z_{i+1}$ for all $0 \leq i \leq nN$.

Lemma 3.3. *Let $u \in \mathrm{Diff}_+(I)$ be a C^{k+1}-flat polynomial diffeomorphism, k, n be positive integers. Then there exists $K > 0$ such that for all sufficiently big $N \geq 1$, and for all $\epsilon : \mathbb{N} \to (0, 1)$ such that $\epsilon(m) \searrow 0$ as $m \to \infty$, if $(z_i)_{0 \leq i \leq nN}$ is an (n, N)-associated sequence of u then there exists a diffeomorphism $g \in \mathrm{Diff}_+^{k+1}(I)$ such that $g(z_i) = z_{i+1}, 0 \leq i \leq nN - 1$, moreover, for all $0 \leq i \leq nN - k - 1, 2 \leq j \leq k + 1$, the following conditions hold:*

(i) $g'(x)$ is monotone on (z_i, z_{i+1}),

(ii) $|g'(z_i) - 1| < \frac{K}{N}$,

(iii) $|g'(z_i) - \overline{u^{(1)}(z_i)}| < \epsilon(N)$,

(iv) $|g^{(j)}(z_i)| < \frac{K}{N}$,

(v) $|g^{(j)}(x)| < 2K \max\{g^{(j)}(z_i), g^{(j)}(z_{i+1})\}$ for all $x \in (z_i, z_{i+1})$.

Proof. First, let us notice that, since u is a C^{k+1}-flat polynomial, for some $K_0 > 0$ independent of N,

$$|\overline{u^{(1)}(z_i)} - 1| < \frac{K_0}{N}, 0 \leq i \leq nN - 1 \text{ and } |\overline{u^{(j)}(z_i)}| < \frac{K_0}{N}, 0 \leq i \leq nN - j.$$

Then condition (ii) follows from condition (iii).

Now, for all $0 \leq i \leq nN-1$, let

$$g'(z_i) = \begin{cases} \overline{u^{(1)}(z_i)} + \epsilon(N) & \text{if } \overline{u^{(1)}(z_i)} < \overline{u^{(1)}(z_{i+1})} \\ \overline{u^{(1)}(z_i)} - \epsilon(N) & \text{if } \overline{u^{(1)}(z_i)} > \overline{u^{(1)}(z_{i+1})} \\ \overline{u^{(1)}(z_i)} & \text{if } \overline{u^{(1)}(z_i)} = \overline{u^{(1)}(z_{i+1})} \end{cases}$$

If $\epsilon(N)$ is sufficiently small then g can be extended such that $g'(x)$ is monotone in the intervals (z_i, z_{i+1}). To satisfy conditions (iv) and (v), it suffices to satisfy $|g^{(k+1)}(x)| < \frac{K}{N}, 0 \leq i \leq nN-k-1, x \in [0,1]$, and this is straightforward. □

The second lemma is perhaps less obvious; we present a detailed proof of it.

Lemma 3.4. *Let n be a positive integer, and $M > 1, r > 0$. Then there exists $K = K(M,n)$ such that for sufficiently big N, if*

(i) $\frac{1}{M} < d_i^{(q)} < M, 1 \leq i \leq nN, 1 \leq j \leq N$,

(ii) $|a_i| < M, 1 \leq i \leq nN-n$,

(iii) $|d_i^q - d_{i'}^{q'}| < \frac{M}{N}$, *for all* $1 \leq i, i' \leq nN, 1 \leq q, q' \leq N$ *such that* $\max\{|i-i'|, |q-q'|\} \leq 1$.

(iv) $|a_i - a_{i+1}| < \frac{M}{N}, 1 \leq i \leq nN-1$,

then there exist $x_1, \ldots, x_{nN}$ *such that* $|x_i| < \frac{K}{N}, 1 \leq i \leq nN$ *and* $|\sum_{q=1}^{N} d_i^{(q)} x_{i-1+q} - a_i| < r, 1 \leq i \leq nN-n$.

Proof. We may and will assume that $M > 2n$. Let us first observe that for arbitrary $x_1, \ldots, x_{nN-1}$ we have

$$|\sum_{q=1}^{N} d_i^{(q)} x_{i-1+q} - \sum_{q=1}^{N} d_{i+1}^{(q)} x_{i+q}| < \sum_{q=2}^{N} |d_i^{(q)} - d_{i+1}^{(q)}||x_{i-1+q}| + |d_i^{(1)} x_i - d_{i+N}^{(N)} x_{i+N}| <$$

$$< \sum_{q=2}^{N} \frac{M}{N}\frac{K_i}{N} + |d_i^{(1)} x_i - d_{i+N}^{(N)} x_{i+N}| < \frac{MK_i}{N} + |d_i^{(1)} x_i - d_{i+N}^{(N)} x_{i+N}| \ (*)$$

where $K_i = N \max\{|x_i|, \ldots, |x_{i+N}|\}$.

Now, let $m = [M^4+1]$ and $N+1 = s_0 < s_1 < \cdots < s_{m-1} < s_m = nN$ such that the quantities $s_{i+1} - s_i, 1 \leq i \leq m-1$ differ from each other by at most one unit.

We will define the sequence $(x_p)_{1\leq i\leq nN}$ inductively on p as follows: first, we choose $x_1 = x_2 = \cdots = x_N \in (0, \frac{M^2}{N}]$ such that $\sum_{q=1}^{N} d_1^{(q)} x_q = a_1$.

For each $p \in [s_i, s_{i+1}), 1 \leq i \leq m-1$, we choose $x_p = \pm\frac{M^{4i}}{N}$ such that $x_p > 0$ if $\sum_{q=1}^{N} d_{p-N}^{(q)} x_{p-N-1+q} < a_{p-N}$, and $x_p < 0$ otherwise.

Thus to guarantee the inequality $|\sum_{q=1}^{N} d_{p-N+1}^{(q)} x_{p-N+q} - a_{p-N+1}| < r$, at each step p, we make an adjustment (when defining x_p) depending if the previous sum is smaller or bigger than needed (i.e. whether or not $\sum_{q=1}^{N} d_{p-N}^{(q)} x_{p-N-1+q} < a_{p-N}$). The size (absolute value) of the adjustment grows as the step p grows, but it is not bigger than $\frac{M^{4m}}{N}$. Similar to the inequality $(*)$ one can show that at every step $p \in [s_i, s_{i+1})$ we have the inequality

$$|\sum_{q=1}^{N} d_{p-N}^{(q)} x_{p-N-1+q} - \sum_{q=1}^{N-1} d_{p-N+1}^{(q)} x_{p-N+q}| < \frac{M^{4i-2}}{N} \quad (*_2).$$

Indeed, we have

$$|\sum_{q=1}^{N} d_{p-N}^{(q)} x_{p-N-1+q} - \sum_{q=1}^{N-1} d_{p-N+1}^{(q)} x_{p-N+q}| \leq d_{p-N}^{(1)}|x_{p-N}| + \sum_{q=1}^{N-1} |d_{p-N}^{(q)} - d_{p-N+1}^{(q)}||x_{p-N+q}|$$

$$\leq d_{p-N}^{(1)}|x_{p-N}| + \sum_{p-N+1+q\leq N} |d_{p-N}^{(q)} - d_{p-N+1}^{(q)}||x_{p-N+q}| +$$

$$\sum_{i+1}^{t} \sum_{p-N+1+q\in[s_i,s_{i+1})} |d_{p-N}^{(q)} - d_{p-N+1}^{(q)}||x_{p-N+q}| \leq$$

$$M\frac{M^2}{N} + \frac{nN}{M^4}\frac{M}{N}(\frac{M^4}{N} + \frac{M^8}{N} + \cdots + \frac{M^{4i}}{N}) = \frac{M^3}{N} + \frac{n}{M^3 N}\frac{M^{4(i+1)}-1}{M^4-1}$$

$$\leq \frac{M^3}{N} + \frac{n}{M^3 N} M^{4i} \leq \frac{M^{4i-2}}{N}.$$

Hence

$$|\sum_{q=1}^{N} d_{p-N}^{(q)} x_{p-N-1+q} - \sum_{q=1}^{N} d_{p-N+1}^{(q)} x_{p-N+q}| \geq \frac{M^{4i-1}}{N} - \frac{M^{4i-2}}{N} > \frac{M^{4i-2}}{N}.$$

Thus we can guarantee the inequality $|\sum_{q=1}^{N} d_i^{(q)} x_{i+q} - a_i| < r$ for all $1 \leq i \leq nN - N$ by satisfying the inequality $|x_1| \leq |x_2| \leq \cdots \leq |x_{nN-1}| \leq \frac{M^{4m}}{N}$. Thus it suffices to take $K = M^{4M^4+4}$. $\square$.

Now we are ready to prove the main lemma about the approximation of roots in the diffeomorphism group $\mathrm{Diff}_+^k(I)$.

Lemma 3.5. *Let* $f \in \mathrm{Diff}_+^k(I)$, $f'(0) = f'(1) = 1, f^{(j)}(0) = f^{(j)}(1) = 0, 2 \leq j \leq k$ *and* $r > 0$. *Let also* f *has no fixed point in (0,1). Then there exists* $N \geq 1$ *and* $g \in \mathrm{Diff}_+^k(I)$ *such that* $||g||_k < r$ *and* $||g^N - f||_k < r$.

Proof. By replacing f with a C^{k+1}-flat polynomial diffeomorphism from its $r/4$ neighborhood, we may assume that $f \in \mathrm{Diff}_+^{k+1}(I)$, and the functions $f'(x) - 1$ and $f^{(j)}(x), 2 \leq j \leq k$ change their sign at most l times. Let $C = 2 + ||f(x) - x||_{k+1}$, n be sufficiently big with respect to l, and $0 < x_0 < x_1 < \cdots < x_n < 1$ such that

$$\max\{x_0, 1 - x_n\} < r/2, f(x_i) = x_{i+1}, 0 \leq i \leq n - 1,$$

moreover, we have

$$\sup_{x \in [0,x_1] \cup [x_{n-1},1]} |f'(x) - 1| < r/2 \text{ and } \sup_{x \in [0,x_1] \cup [x_{n-1},1], 2 \leq j \leq k+1} |f^{(j)}(x)| < r/2.$$

Let $N \geq 1$ be sufficiently big with respect to $\max\{n, C, l\}$, and $z_0, z_1, z_2, \ \ldots, z_{nN}$ be a sequence such that $z_{iN} = x_i, 0 \leq i \leq n, z_k = x_0 + \frac{x_1 - x_0}{N}k, 0 \leq k \leq N$, and $z_{iN+j} = f(z_{iN-N+j}), 1 \leq i \leq n-1, 1 \leq j \leq N$. The subsequence $(z_1, z_2, \ldots, z_{N-1})$ will be called *the initial subsequence*; it determines the entire sequence $(z_i)_{0 \leq i \leq nN}$.

We define the diffeomorphism $g \in \mathrm{Diff}_+(I)$ as follows. Firstly, we let $g(z_i) = z_{i+1}, 0 \leq i \leq nN - 1$. Let us observe that we already have $g^N(z_i) = z_{i+N} = f(z_i), 0 \leq i \leq nN - N$. Notice that as $N \to \infty$ the quantity $\max_{0 \leq i \leq N-1} \frac{z_{i+2} - z_{i+1}}{z_{i+1} - zi}$ tends to 1, by Mean Value Theorem. Hence, we have $||g^N(x) - f(x)||_0 < r$ **(1)** (i.e. for all possible extensions of g).

Now we need to define $(g^N)^{(j)}(z_i)$, $0 \leq i \leq nN - N, 1 \leq j \leq k$ such that for a suitable extension of g we could still claim that $||g^N(x) - f(x)||_k < r$. (Obviously, the latter cannot hold for an arbitrary extension of g.) We also need to make sure that g will be small (i.e. close to the identity) in the C^k norm.

By chain rule, to define the quantities $(g^N)^{(j)}(z_i),\ 0 \le i \le nN - N, 1 \le j \le k$, it suffices to define the values $g^{(j)}(z_i),\ 0 \le i \le nN - N, 1 \le j \le k$ of the derivatives of the function g up to the k-th order. Let us observe that for all $0 \le i \le nN - N$, we have

$$(g^N)''(z_i) = g''(z_i)\frac{\Pi}{g'(z_i)} + g''(z_{i+1})\frac{\Pi g'(z_i)}{g'(z_{i+1})} + g''(z_{i+2})\frac{\Pi g'(z_i)g'(z_{i+1})}{g'(z_{i+2})} + \\ + \dots + g''(z_{i+N-1})\frac{\Pi g'(z_i)g'(z_{i+1})\dots g'(z_{i+N-2})}{g'(z_{i+N-1})} \quad (*)$$

where $\Pi = g'(z_i)g'(z_{i+1})\dots g'(z_{i+N-1})$.

The right-hand side of the formula $(*)$ defines a polynomial P_2 (independent of i) of $2N$ variables [3]
such that

$$(g^N)''(z_i) = P_2(g'(z_i), \dots, g'(z_{i+N-1}), g''(z_i), \dots, g''(z_{i+N-1})), 0 \le i \le nN - N.$$

Let us also observe that if g is of class $C^j, j \ge 3$ then

$$(g^N)^{(j)}(z_i) = g^{(j)}(z_i)\frac{\Pi}{g'(z_i)} + \dots + g^{(j)}(z_{i+N-1})\frac{\Pi g'(z_i)g'(z_{i+1})\dots g'(z_{i+N-2})}{g'(z_{i+N-1})} + A_1 + \dots + A_d \quad (*_j)$$

where $d \le N^{2j}$ for sufficiently big N, and each of the terms $A_p, 1 \le p \le d$ consists of a product of at least $N-1$ and at most N^j derivatives $g^{(j-l)}(z_{i+s}), 0 < l \le j-1, 0 \le s \le N-1$, moreover, at least two and at most j of the derivatives in the product are values of derivatives of order $j - l \in \{2, \dots, j-1\}$.

Again, the right-hand side of the formula $(*_j)$ defines a polynomial P_j of jN variables such that

$$(g^N)^{(j)}(z_i) = P_j(g'(z_i), \dots, g'(z_{i+N-1}), \dots, g^{(j)}(z_i), \dots, g^{(j)}(z_{i+N-1})), 0 \le i \le nN - N.$$

[3] More precisely, the polynomial P_2 is given by the formula

$$P_2(X_1, \dots, X_N, Y_1, \dots, Y_N) = R_1(X_1, \dots, X_N)Y_1 + \dots + R_N(X_1, \dots, X_N)Y_N$$

where $R_1, \dots, R_N$ are polynomials in N variables $X_1, \dots, X_N$ defined as follows:

$$R_1(X_1, \dots, X_N) = X_2X_3\dots X_N = \frac{\omega}{X_1},$$

$$R_2(X_1, \dots, X_N) = X_1^2X_3\dots X_N = \frac{\omega X_1}{X_2},$$

$$R_3(X_1, \dots, X_N) = X_1^2X_2^2X_4\dots X_N = \frac{\omega X_1X_2}{X_3}, \dots,$$

$$R_N(X_1, \dots, X_N) = X_1^2X_2^2\dots X_{N-1}^2 = \frac{\omega X_1X_2\dots X_{N-1}}{X_N}.$$

Here, $\omega = X_1 \dots X_N$.

We make an important observation that

$$P_j(g'(z_i), \ldots, g'(z_{i+N-1}), \ldots, g^{(j)}(z_i), \ldots, g^{(j)}(z_{i+N-1})) =$$

$$= P_2(g'(z_i), \ldots, g'(z_{i+N-1}), g^{(j)}(z_i), \ldots, g^{(j)}(z_{i+N-1})) + A_1 + \cdots + A_d \ (*)$$

Now, we would like to introduce the quantities $\overline{g^{(j)}(z_i)}, 0 \leq j \leq k+1, 0 \leq i \leq nN - k$ by defining them inductively on j as follows:

$$\overline{g^{(j+1)}(z_i)} = \frac{\overline{g^{(j)}(z_{i+1})} - \overline{g^{(j)}(z_i)}}{z_{i+1} - z_i}$$

and $\overline{g^{(0)}(z_i)} = g(z_i)$.

Then, using the formula $(*_j)$ and Lemma 3.4, inductively on j, for sufficiently big N, we can choose a constant $K = K(C, n, r, l) > C^{k+1}$, the values $g^{(j)}(z_i), 0 \leq i \leq nN - N$ such that the following conditions hold

(c-i) $|\overline{g'(z_i)} - 1| < \frac{K}{N}$,

(c-ii) $|\overline{g^{(j)}(z_i)}| < \frac{K}{N}$ for all $2 \leq j \leq k+1$,

(c-iii) the sequence $\overline{g'(z_i)}, \ldots, \overline{g'(z_{i+N})}$ is l-quasi-monotone;

(c-iv) $\frac{1}{K}|\overline{g^{(j)}(z_i)}| < |g^{(j)}(z_i)| < K|\overline{g^{(j)}(z_i)}|$ for all $1 \leq j \leq k+1$,

(c-v) $|P_j(g'(z_i), \ldots, g'(z_{i+N-1}), \ldots, g^{(j)}(z_i), \ldots, g^{(j)}(z_{i+N-1})) - f^{(j)}(z_i)| < \frac{r}{2}$

(c-vi) $g'(z_i) = \frac{z_{i+2} - z_{i+1}}{z_{i+1} - z_i} + \epsilon_i^{(N)}, 0 \leq i \leq nN - 2$, where $\mathrm{sgn}(\epsilon_i^{(N)}) = \mathrm{sgn}(\overline{g'(z_i)} - \overline{g'(z_{i+1})})$.

(c-vii) the sequences $(g'(z_i), \ldots, g'(z_{i+N}))$ are l-quasi-monotone.

Indeed, conditions (c-i) and (c-ii) for the sequence $\overline{g^{(j)}(z_i)}, 1 \leq i \leq nN$ for all $1 \leq j \leq k+1$, follow from the fact that all of the functions $(f^{(j)})^s, 0 \leq j \leq k, 1 \leq s \leq n$ are polynomials (i.e. all n iterates of the function $f^{(j)}$ for all $0 \leq j \leq k$). By the choice of f and by the Mean Value Theorem, the monotonicity of the sequence $(\overline{g'(z_i)}, \ldots, \overline{g'(z_{i+N})})$ is violated in at most l terms.

After this, for all $2 \leq j \leq k$, let

$$d_{i,j}^{(q)} = R_q(g'(x_i), \ldots, g'(x_{i+N-1}), g^{(j)}(x_i), \ldots, g^{(j)}(x_{i+N-1}).$$

Let also

$$a_i = P_2(g'(x_i), \ldots, g'(x_{i+N-1}), g^{(j)}(x_i), \ldots, g^{(j)}(x_{i+N-1})),$$

i.e. we have

$$a_i = P_j(g'(x_i), \dots, g'(x_{i+N-1}), g''(x_i), \dots, g''(x_{i+N-1}), \dots, g^{(j)}(x_i), \dots, g^{(j)}(x_{i+N-1})$$

$$-A_1 - \cdots - A_d.$$

Then for each fixed $j \in \{2, \dots, k\}$, the quantities $d_i^{(q)} := d_{i,j}^{(q)}$ satisfy the conditions of Lemma 3.4 for some positive constants M and $K(M, l, r, n)$ where the latter is independent of N. By this lemma and by Lemma 3.3, using the formulas $(*_j)$, inductively on j, we can guarantee the conditions (c-iv) and (c-v). By choosing $\epsilon^{(N)}$ sufficiently small and defining $g'(z_i), 1 \leq i \leq nN$ as in condition (c-vi), by the l-quasi-monotonicity of $\overline{g'(z_i)}, 1 \leq i \leq nN$ we also obtain the l-quasi-monotonicity of $g'(z_i), 1 \leq i \leq nN$, which is the condition (c-vii) (i.e. the monotonicity of the sequence $(\overline{g'(z_i)}, \dots, \overline{g'(z_{i+N})})$ is violated in at most l terms).

For simplicity, let us assume that the sequences $(g'(z_i), \dots, g'(z_{i+N}))$ are quasi-monotone, i.e. for some $p \in \{i, \dots, i+N-1\}$, both of the subsequences $(g'(z_i), \dots, g'(z_{i+N}))$ and $(g'(z_{i+p}), \dots, g'(z_{i+N}))$ are monotone (again, the general case is similar).

Now we need to extend the definition of $g(x)$ to all $x \in [0,1]$ such that we have

$$|g'(x) - 1| < r(\mathbf{2}), |g^{(s)}(x)| < r\ (\mathbf{3}) \text{ and } |(g^N)^{(j)}(x) - f^{(j)}(x)| < r\ (\mathbf{4})$$

for all $x \in [0,1]$ and $2 \leq s \leq k, 1 \leq j \leq k$. By condition (c-v), for sufficiently big N, and sufficiently small $\epsilon^{(N)} := \max\limits_{1 \leq i \leq N} |\epsilon_i^{(N)}|$, we can extend the function g at each of the intervals (z_i, z_{i+1}) such that $g'(z)$ is monotone on (z_i, z_{i+1}). Also, by Lemma 3.3 and by Mean Value Theorem, the extension can also be made to satisfy

$$|g^{(j)}(z)| < 2K \max\{g^{(j)}(z_i), g^{(j)}(z_{i+1})\}\ (\mathbf{5})$$

and

$$|g^{(j)}(z) - g^{(j)}(z_i)| < \frac{2K^2}{N}|z_{i+1} - z_i|(\mathbf{6})$$

for all $2 \leq j \leq k$.

By monotonicity of $g'(z)$ on each subinterval (z_i, z_{i+1}) and by condition (c-i) we obtain the inequality **(2)**. From the inequality **(5)** and condition (c-ii), we obtain the inequality **(3)**.

For the inequality **(4)**, let us first point out that, by condition (c-iv), we have this inequality for each value $z = z_i, 0 \leq i \leq nN-k$. To obtain it for all possible values it suffices to show that

$$|(g^N)^{(j)}(z) - (g^N)^{(j)}(z_i)| < r/2 \ (\mathbf{7})$$

for all $z \in (z_i, z_{i+1})$.

For simplicity, we will first prove the inequality **(7)** for $j = 2$. (for $j = 1$, it is already proved in the proof of Lemma 3.2 in [2]). The right-hand side of the formula $(*)$ also defines monomials $R_1, \dots, R_N$ on N variables such that we can write

$$(g^N)''(z_i) = g''(z_i)R_1(g'(z_i), \dots, g'(z_{i+N-1})) + g''(z_{i+1})R_2(g'(z_i), \dots, g'(z_{i+N-1})) +$$

$$+ \dots + g''(z_{i+N-1})R_N(g'(z_i), \dots, g'(z_{i+N-1})).$$

[4]

For $z \in (z_i, z_{i+1})$, we can also write

$$(g^N)''(z) = g''(t_i)R_1(g'(t_i), \dots, g'(t_{i+N-1})) + g''(t_{i+1})R_2(g'(t_i), \dots, g'(t_{i+N-1})) +$$

$$+ \dots + g''(t_{i+N-1})R_N(g'(t_i), \dots, g'(t_{i+N-1}))$$

where $t_m \in (z_m, z_{m+1})$ for all $m \in \{i, i+1, \dots, i+N-1\}$.

Then, taking into account that $(1 + \frac{K}{N})^N \nearrow e^K$ we obtain

$$|(g^N)''(z_i) - (g^N)''(z)| \leq \sum_{1 \leq m \leq N} R_{m+1}(g'(z_i), \dots, g'(z_{i+N-1}))|g''(z_{i-1+m}) - g''(t_{i-1+m})| +$$

$$+ \sum_{1 \leq m \leq N} |R_{m+1}(g'(z_i), \dots, g'(z_{i+N-1})) - R_{m+1}(g'(t_i), \dots, g'(t_{i+N-1}))| \cdot |g''(t_{i-1+m})| \leq$$

$$\leq \sum_{1 \leq m \leq N} R_{m+1}(g'(z_i), \dots, g'(z_{i+N-1}))|z_{i+m} - z_{i-1+m}| \frac{2K^2}{N} +$$

$$+ \sum_{1 \leq m \leq N} |R_{m+1}(g'(z_i), \dots, g'(z_{i+N-1})) - R_{m+1}(g'(t_i), \dots, g'(t_{i+N-1}))| \frac{2K^2}{N}$$

$$\leq e^{2K} \frac{2K^3}{N} + \sum_{1 \leq m \leq N} C^2 |z_{i+m} - z_{i-1+m}| \frac{2K^2}{N} \leq 4e^{2K} \frac{K^3}{N}$$

---

[4]Let us recall that we have

$$R_1(u_1, \dots, u_N) = u_2 u_3 \dots u_N, \ R_2(u_1, \dots, u_N) = u_1^2 u_3 u_4 \dots u_N,$$

$$R_3(u_1, \dots, u_N) = u_1^2 u_2^2 u_4 u_5 \dots u_N$$

and so on.

Thus we proved the inequality **(7)** for $j = 2$. The proof for a general j is similar. Let us call a monomial *special* if it is of the form $R(u_1, \dots, u_N) = u_p u_{p+1} \dots u_q$ for some $1 \leq p \leq q \leq N$. Then we can write

$$(g^N)^{(j)}(z_i) = g^{(j)}(z_i)R_1(g'(z_i), \dots, g'(z_{i+N-1})) + \dots +$$

$$g^{(j)}(z_{i+N-1})R_N(g'(z_i), \dots, g'(z_{i+N-1})) + \sum_{\alpha \in S} B_\alpha R_\alpha(g'(z_i), \dots, g'(z_{i+N-1}))$$

where $|S| = d$, and the terms $B_\alpha R_\alpha(g'(z_i), \dots, g'(z_{i+N-1})) = A_\alpha, 1 \leq \alpha \leq d$ are the terms $A_1, \dots, A_d$ from the formula $*$). Thus each B_α is a product of at least two and at most j derivatives $g^{(j-l)}(z_{i+s})$ where $1 \leq l \leq j-2, 0 \leq s \leq N-1$, and R_α is a product of at most $2j$ special monomials. Furthermore, the terms have the following finer properties: for each $2 \leq q \leq j$, let C_q denotes the set of ordered q-tuples $\omega = (k_1, \dots, k_q)$ (with repetitions allowed) of the set $\{i, i+1, \dots, i+N-1\}$. Then one can partition $S = \sqcup_{1 \leq q \leq j} D_q$ such that $|D_q| \leq (jN)^q |C_q|$ with a surjection $\xi_q : D_q \to C_q, 1 \leq q \leq j$ where if $\alpha \in D_q$ and $\xi(\alpha) = (k_1, \dots, k_q)$ then the term B_α consists of the product $g^{(j_1)}(x_{i+k_1}) \dots g^{(j_q)}(x_{i+k_q})$ for some $j_1, \dots, j_q \in \{2, \dots, j\}$, moreover, for all $x \in C_q$ we have $|\xi_q^{-1}(x)| \leq (jN)^q$. We let

$$a_i = \sum_{\alpha \in S} A_\alpha R_\alpha(g'(z_i), \dots, g'(z_{i+N-1}))$$

for all $0 \leq i \leq nN - n$. By properties (c-i), (c-ii) and (c-iv), the quantities $d_{i,j}^{(q)}, 1 \leq i \leq nN - n, 2 \leq j \leq k, 1 \leq q \leq N$ and $a_i, 0 \leq i \leq nN - n$ satisfy the conditions of Lemma 3.4. Then, by estimating the difference $|(g^N)^{(j)}(z_i) - (g^N)^{(j)}(z)|$ we can derive the inequality similarly to the case of $j = 2$ $\square$

Now we would like to quote a perfectness result from [18] which we have used in Section 2.

Lemma 3.6. *For all* $f \in \mathrm{Diff}_c^\infty(I)$ *there exists* $u_1, \dots, u_8 \in \mathrm{Diff}_c^\infty(I)$ *such that* $f = [u_1, u_2][u_3, u_4][u_5, u_6][u_7, u_8]$.

Proof. See [18].

Lemma 3.5 and Lemma 3.6 are the results we needed for the proof of Theorem 1.1. Now we need to extend these results to higher dimensions. Let $d \geq 2$, Ω be an open unit ball in $\mathbb{R}^d$, and $I^d = (0,1)^d$ be an open unit cube in $\mathbb{R}^d$. We would like to introduce a key notion of an *elementary diffeomorphism*.

Definition 3.7. A diffeomorphism $f \in \mathrm{Diff}_c^{\infty}(\Omega)$ is called an elementary diffeomorphism if there exists $\phi : \overline{I^d}$ to $\overline{\Omega}$ such that ϕ is a restriction of a C^{∞} diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$, and for all $(x,t) \in I^{d-1} \times I$ we have $\phi^{-1} f \phi(x,t) = (x, \psi_x(t))$ where for all $x \in I^{d-1}$, the support of the diffeomorphism $\psi_x \in C_c^{\infty}(I)$ is connected. More generally, for $k \in \{1, \dots, d-1\}$, we say f is a k-elementary diffeomorphism if there exists $\phi : \overline{I^d} \to \overline{\Omega}$ as a restriction of a C^{∞} diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $(x,u) \in I^{d-k} \times I^k$ we have $\phi^{-1} f \phi_(x,u) = (x, \psi_x(u))$ where, for all $x \in I^{d-k}$, the support of the diffeomorphism $\psi_x \in C_c^{\infty}(I)$ is connected.

Proposition 3.8. *Let Ω be an open unit ball in $\mathbb{R}^d$, $r > 0$ and $g \in \mathrm{Diff}_c^k(\Omega)$ where k is a positive integer. Then there exists a diffeomorphism $h \in \mathrm{Diff}_c^k(\Omega)$ such that $h \in B_r(g)$ and h is a product of elementary diffeomorphisms.*

Proof. Let S be unit sphere i.e. the boundary of Ω, and $\tilde{S}$ be the quotient of S obtained by identifying the antipodal points. For every unit vector $\mathbf{x} \in \mathbb{R}^d$, we will write $\mathcal{F}_{\mathbf{x}}$ to denote the foliation of Ω with straight segments parallel to $\mathbf{x}$. A diffeomorphism $f \in \mathrm{Diff}_c^k(\Omega)$ maps the foliation $\mathcal{F}_{\mathbf{x}} = \{L_z\}_{z \in \tilde{S}}$ to a foliation $f\mathcal{F}_{\mathbf{x}} = \{f(L_z)\}_{z \in \tilde{S}}$ where for all $z \in \tilde{S}$, the leaf $f(L_z)$ shares the same ends with L_z.

We will fix a unit vector $\mathbf{x_0}$. For simplicity, let us assume that $d = 2$. We will also assume that the interior of the support of f is diffeomorphic to an open ball. Then the leaves L_z and $f(L_z)$ bound a unique domain in Ω which we will denote by D_z.

An arc $C = (p,q)$ of a leave $f(L)$ of the foliation $f\mathcal{F}_{\mathbf{x_0}}$ will be called *maximal* if C is *convex*, i.e. any segment connecting any two points on C is totally inside the domain bounded by L and $f(L)$, moreover, C is not properly contained in any other convex arc.

Let $\mathcal{A}_f$ denotes the set of all maximal arcs of $f\mathcal{F}_{\mathbf{x_0}}$. An element $a \in \mathcal{A}_f$ belongs to a certain leaf L_z, and will also write $D(a)$ to denote the domain D_z. We will introduce an equivalence relation on $\mathcal{A}_f$ as follows: the elements $a_1, a_2 \in \mathcal{A}_f$ are called equivalent if there exists a continuous family of maximal arcs starting at a_1 and ending at a_2. The set of equivalence classes of $\mathcal{A}_f$ will be denoted with A_f. For any $a \in \mathcal{A}_f$ the union of all maximal arcs which can be reached from a by a continuous family of maximal arcs will be denoted by $\Omega(a)$.

By assumption, the support of f is diffeomorphic to a ball. This allows us to introduce a partial order in the set A_f. Let $\alpha_1, \alpha_2 \in A_f$.

We say α_1 *dominates* α_2 if for any $\epsilon > 0$ there exist representatives $a_1, a_2 \in \mathcal{A}_f$ of α_1, α_2 respectively such that the arcs a_1, a_2 are ϵ-close to each other, and a_2 belongs to the domain $D(a_1)$. Then, we say $\alpha_2 \prec \alpha_1$ if there exists a sequence $(\beta_1, \dots, \beta_n)$ such that β_{i+1} dominates β_i for all $0 \leq i \leq n-1$, and $\beta_0 = \alpha_2, \beta_n = \alpha_1$.

Let us notice that if a diffeomorphism $f \in \mathrm{Diff}_c^k(\Omega)$ acts along the leaves of $\mathcal{F}_{\mathbf{x}}$ then it is an elementary diffeomorphism provided that the interior of the support is diffeomorphic to a ball.

Now we choose f from the neighborhood $B_{r/2}(g)$ in the space $\mathrm{Diff}_c^k(\Omega)$ such that supp$f$ is diffeomorphic to a ball and the restriction of f to this support coincides with a polynomial diffeomorphism. Then the set A_f is finite. Let $n = |A_f|$ and α be a maximal element of A_f. Then there exists an elementary diffeomorphism h_1 such that $\mathrm{supp}(h_1) \subseteq \Omega(\alpha)$ and $|A_{h_1 f}| < n$. Continuing the process inductively we find elementary diffeomorphisms $h_1, \dots, h_{n-1}$ such that the diffeomorphism $h_1 \dots h_{n-1} f$ has only one equivalence class of maximal arcs, i.e. $|A_{h_1 \dots h_{n-1} f}| = 1$. Then it is elementary. Thus f is a product of elementary diffeomorphisms.

The case of higher dimensions $d > 2$ is similar. We replace the concept of maximal arc with a concept of maximal k-discs. Then we first show that g can be approximated by a product of finitely many $(d-1)$-elementary diffemorphisms. Then we can approximate each of these $(d-1)$-elementary diffemorphisms with a product of finitely many $(d-2)$-elementary diffemorphisms, and proceed by induction. □

The following lemma follows from the proof of Lemma 3.5 as it is straightforward to extend the construction there (for the interval I) smoothly in the vertical direction of the cube $I^d = I \times I^{d-1}$.

Lemma 3.9. *Let Ω be an open unit ball in $\mathbb{R}^d$, $r > 0$ and $F \in \mathrm{Diff}_c^k(\Omega)$ be an elementary diffeomorphism. Then there exists $N \geq 1$ and $g \in \mathrm{Diff}_c^k(\Omega)$ such that $||g||_k < r$ and $||g^N - F||_k < r$.*

Proof. Indeed, in the proof of Lemma 3.5, By replacing F with another diffeomorphism from its $r/4$-neghborhood if necessary, we may assume that there exists N_0 such that for all $x \in I^{d-1}$ and for all diffeomorphisms $F_x(t) = F(x,t)$ of the interval I the number N from the proof of Lemma 3.5 can be taken equal to N_0. We will fix this N_0 for the rest of the proof.

We also observe that the construction of the N_0-th root can be made continuous in f, i.e. for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if f_1 is a C^k-flat diffeomorphism of I and $||f_1 - f||_k < \delta$, then one can construct a diffeomorphism g_1 such that $||g_1 - g||_k < \epsilon$ and $||g_1^{N_0} - f_1||_k < r/4$.

Now, let $\epsilon_0 > 0$ be such that if $||g - g_1||_k < \epsilon_0$ then $||g^{N_0} - g_1^{N_0}||_k < r/4$. Let also $\delta_0 = \delta(\epsilon_0)$. We can and will also assume that $\epsilon_0 < r/4$.

Let $\delta_1 > 0$ be such that if $x, y \in I^{d-1}$ are at a distance at most δ_1 apart, in the l^1-norm $||.||$ of $\mathbb{R}^{d-1}$, then $||F_x - F_y||_k < \delta_0$. Let $S = \frac{1}{m}\mathbb{Z}^{d-1} \cap I^{d-1}$ where m is big enough that S is a $\frac{1}{2}\delta_1$-net in I^{d-1} (in the l^1-norm $||.||$ of $\mathbb{R}^{d-1}$). For all $x \in S$, we can choose a diffeomorphism $g_x(t)$ from an ϵ_0-neighborhood of the identity in the space $C^k(I)$ such that $||g_x^{N_0} - F_x||_k < r/4$; moreover, for all $x, y \in S$, if the distance $||x - y||$ is less than δ_1 then $||g_x - g_y|| < \epsilon_0$.

We need to define the diffeomorphisms g_x for all $x \in I^{d-1}$ (not just for $x \in S$). But for all $x, y \in S$, if the distance $||x - y||$ is less than δ_1 then, in the space $C^k(I)$, the diffeomorphisms g_x and g_y can be connected by a path in the ball of radius ϵ_0 centered at the origin. Thus we can extend a finite collection $\{g_x\}_{x\in S}$ to a diffeomorphism $g \in \mathrm{Diff}_c^k(I \times I^{d-1})$ such that $||g||_k < 4\epsilon_0$ and $||g^{N_0} - F||_k < r$. □

As for the perfectness results, we quote the following theorem from the literature. We have used only the perfectness claim in this theorem (but not the uniform perfectness with the given strong bound) which has been proved in a number of sources mentioned in the introduction.

Lemma 3.10. *Let M be a compact smooth manifold. Then the group $\mathrm{Diff}_c^\infty(M)_0$ is perfect. Moreover, there exists an open neighborhood U of identity such that for all $f \in \mathrm{Diff}_c^\infty(M)$ there exist $u_1, \dots, u_8 \in \mathrm{Diff}_c^\infty(M)$ such that $f = [u_1, u_2][u_3, u_4][u_5, u_6][u_7, u_8]$.*

Proof. See [10].

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