NAZAROV (CBMS)

Problem $0 \le \varphi$, $\varphi \in C_0^\infty$,

$s \in [0, n-2)$,

(log @ s=0) $$U = \varphi * \frac{1}{|x|^s} = \int \frac{1}{|x-y|^s}\varphi(y)\,dm_n(y).$$

$$\sup_{x \in \mathbb{R}^n} |\nabla U(x)| \le C \sup_{x \in \operatorname{supp}(\varphi)} |\nabla U(x)| \, ?$$

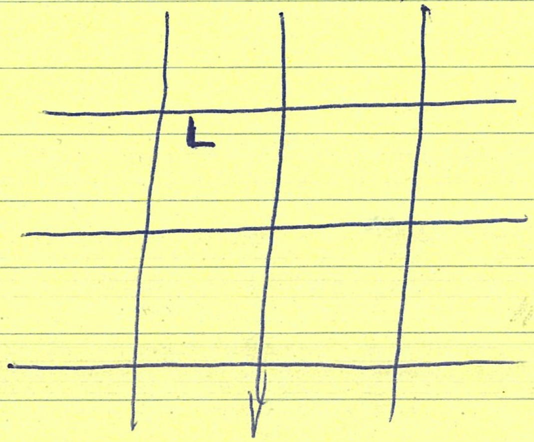
Uniformly Disconnected Sets

A set $F \subset \mathbb{R}^n$ is uniformly disconnected if $\forall x \in F$, $\ell > 0$, F can be written as a disjoint union $F = F_x \cup F^x$ s.t.

$x \in F_x$, $\operatorname{diam}(F_x) \approx \ell$, $\operatorname{dist}(F_x, F^x) \gtrsim \ell$.

Dyadic structure (Cantor structure)

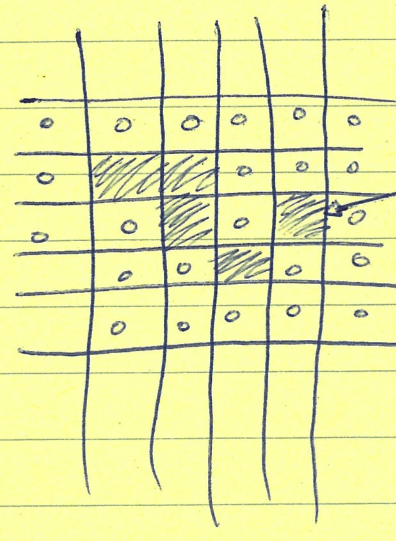
Suppose that F is UD. Fix $L > 0$,



L-grid.

Consider those cubes that intersect L. Since F is UD, any chain of adjacent cubes intersecting F has bounded

Cardinality.



Q an elementary cell.

If Q', Q'' are disjoint, then $d(Q', Q'') \gtrsim \max(\ell(Q'), \ell(Q''))$

$\ell(Q) \to$ sidelength of a dyadic cube ~~in the~~ generating Q.

<u>FACT.</u> If we have a sequence of Cantor structures, there is a subsequence that stabilizes at every level.

<u>Singular Integral Operators</u>

$$K : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Let μ be a locally-finite non-negative Borel measure on $\mathbb{R}^n$ with UD support, satisfying

- $\forall \delta > 0, \ \sup_{|x-y|>\delta} |K(x,y)| < +\infty$,
- $K(x,y) = -K(y,x)$.

Formally, $$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, d\mu(y).$$

<u>Simple Functions</u>. Finite linear combinations of (pairwise disjoint) cells. Closed under addition.

$$f = \sum_Q a_Q \chi_Q, \quad g = \sum_Q b_Q \chi_Q.$$

(may assume cubes are from the same level).

$$\langle T_\mu f, g \rangle_\mu = \sum_{Q' \neq Q} a_Q b_{Q'} \iint_{Q \times Q'} K(x,y)\, d\mu(x)\, d\mu(y)$$

Doesn't depend on the regularization.

<u>Def$^{\underline{n}}$</u>. T_μ is <u>bounded</u> in $L^2(\mu)$ if

$$|\langle T_\mu f, g \rangle_\mu| \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}$$

$$\forall \text{ simple } f, g.$$

<u>Restriction</u>. Suppose μ has UD support, consider some sequence Q_j of pairwise disjoint cells.

$$\mu = \mu' \chi_{\underbrace{\mathbb{R}^n \setminus \bigcup_j Q_j}_{E}}.$$

For f, g simple, and provided that T_μ is bounded in $L^2(\mu)$,

$$\langle T_{\mu'} f, g \rangle_{\mu'} = \langle T_\mu (f \chi_E), g \chi_E \rangle_\mu$$

(Extension is anti-symmetric.)

<u>Wolff's potential</u>

Let μ be a locally finite measure.

Fix $s \in (0, n)$, $p > 0$.

$$W_{p,s}[\mu](x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^s} \right)^p \frac{dr}{r}$$

$$\mathcal{W}_{p,s}(\mu) = \int_{\mathbb{R}^n} W_{p,s}[\mu](x)\, d\mu(x).$$

If μ is UD,

$$W_{p,s}[\mu](x) \approx \sum_{Q : x \in Q} \Big(\overbrace{\frac{\mu(Q)}{\ell(Q)^s}}^{D(Q)} \Big)^p, \text{ and,}$$

$$\mathcal{W}_{p,s}(\mu) \approx \sum_Q \left(\frac{\mu(Q)}{\ell(Q)^s} \right)^p \mu(Q).$$

$$\approx \sum_Q D(Q)^p \mu(Q).$$

Reguera - Tolsa Theorem.

$K(x,y) = \frac{x-y}{|x-y|^{s+1}}$, μ has UD support, satisfying $D(Q) \leq 1 \;\forall Q$ T.F.A.E.

(1) T_μ is bounded in $L^2(\mu)$,

(2) $\forall Q$, $\mathcal{W}_{2,s}(\chi_Q \mu) \leq C\mu(Q)$.

True for $s \in [d-1, d)$. (Reguera - Tolsa)
$s \in (0,1]$ (Mateu - Prat - Verdera)

Intermediate statement: T(1) - theorem.

The Carleson Embedding Theorem.

Let μ be a measure (with UD support),
let $\{a_Q\}_Q$ be any family of non-negative numbers, indexed by the Cantor cells.
For $f \in L^1_{loc}(\mu)$, put

$$\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f \, d\mu.$$

Then T.F.A.E.

(1) $\sum_Q a_Q \langle f \rangle_Q^2 \leq C\|f\|^2_{L^2(\mu)}$,

(2) $\sum_{Q \subset P} a_Q \leq C\mu(P)$.