Definition 2.5: The space of homogeneous harmonics of degree n in d dimensions, $\mathbb{Y}_n(\mathbb{R}^d)$. (All homogeneous polynomials $f(\vec{x})$ of degree n in $\mathbb{R}^d$ s.t. $\Delta f(\vec{x}) = 0 \Rightarrow (\nabla \cdot \nabla) f(x) = \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j}\right)^2 f(x) = 0.$)

$\boxed{2.1.3}$ Spherical Harmonics

Definition 2.7: $\mathbb{Y}_n^d := \mathbb{Y}_n(\mathbb{R}^d)\Big|_{S^{d-1}}$ is called the spherical harmonic space of order n in d dimensions.

Remark: $\forall\ Y_n \in \mathbb{Y}_n^d$, there is some $H_n \in \mathbb{Y}_n(\mathbb{R}^d)$ s.t. $H_n(r\xi) = r^n Y_n(\xi)$.

Hence, $\dim(\mathbb{Y}_n^d) = \dim(\mathbb{Y}_n(\mathbb{R}^d))$ ($= N_{n,d}$ as given by (2.10))

Remark: Let $\xi \in S^{d-1}$ be fixed. A function $f: S^{d-1} \to \mathbb{C}$ is said to be invariant with respect to $\mathbb{O}^d(\xi)$ if

$$f(A\eta) = f(\eta) \quad \forall A \in \mathbb{O}^d(\xi) \text{ and } \eta \in S^{d-1}.$$

Also, note that $\mathbb{O}^d(\xi) = \{A \in \mathbb{O}^d : A\xi = \xi\}$. (Note: $\mathbb{O}^d = \{A \in \mathbb{R}^{d \times d} : A^T A = I\}$)

Theorem 2.8: Let $Y_n \in \mathbb{Y}_n^d$ and $\xi \in S^{d-1}$. Then Y_n is invariant w.r.t. $\mathbb{O}^d(\xi)$ if and only if $Y_n(\eta) = Y_n(\xi)\, P_{n,d}(\xi \cdot \eta) \quad \forall \eta \in S^{d-1}$.

Remark: Here, $P_{n,d}$ represents the restriction of the Legendre harmonic on the unit sphere. That is, $P_{n,d}(t) = L_{n,d}\Big|_{S^{d-1}}(\xi_{(d)})$

Also, $L_{n,d} \in \mathbb{Y}_n(\mathbb{R}^d)$ satisfies $L_{n,d}(A\vec{x}) = L_{n,d}(\vec{x}) \ \forall A \in \mathbb{O}^d(e_d), \vec{x} \in \mathbb{R}^d$ and $L_{n,d}(e_d) = 1$.

by definition. Here, $e_d = (\underbrace{0, \ldots 0}_{d-1}, 1)$.

$\xi_{(d)} = t\, e_d + (1-t^2)^{1/2} \xi_{(d-1)}$

2.2 Addition Theorem and Its Consequences

Theorem 2.9: (Addition Theorem) Let $\{Y_{n,j} : 1 \le j \le N_{n,d}\}$ be an orthonormal basis of $\mathbb{Y}_n^d$. That is,

$$\int_{S^{d-1}} Y_{n,j}(\eta)\, \overline{Y_{n,k}(\eta)}\, dS^{d-1}(\eta) = \delta_{jk} \quad \text{where } 1 \le j,k \le N_{n,d}.$$

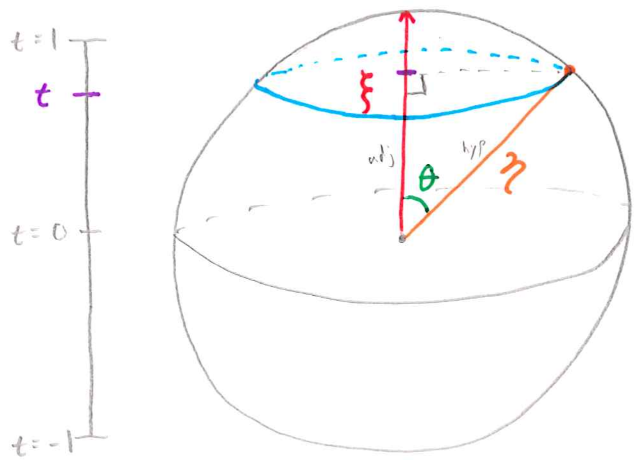
Then we have that

$$\sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\, \overline{Y_{n,j}(\eta)} = \frac{N_{n,d}}{|S^{d-1}|} P_{n,d}(\xi \cdot \eta) \quad \forall\, \xi, \eta \in S^{d-1}.$$

Remark: Recall that $|S^{d-1}| := \int_{S^{d-1}} dS^{d-1}$, and $N_{n,d} = \dim(\mathbb{Y}_n(\mathbb{R}^d))$

Remark: Interpreting the Theorem: $\left(\frac{1}{N_{n,d}}\right) \sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\, \overline{Y_{n,k}(\eta)} = \left(\frac{1}{|S^{d-1}|}\right) P_{n,d}(\xi \cdot \eta).$

Consider the RHS. What exactly is $P_{n,d}(\xi \cdot \eta)$?
Recall that $P_{n,d}(t) := L_{n,d}(\xi_{(d)})$ where $\xi_{(d)} = t e_d + (1-t^2)^{1/2} \xi_{(d-1)}$.



In our case $t = \xi \cdot \eta = \|\xi\| \|\eta\| \cos\theta$
$= \cos\theta$ since $\xi, \eta \in S^{d-1}$
$= \frac{\text{adj}}{\text{hyp}} = \frac{\text{adj}}{1}$ since $\|\eta\| = \|\text{hyp}\|$.

Since $L_{n,d}(\xi_{(d)})$ is invariant under rotations which fix e_d, then $P_{n,d}(\alpha)$ is constant along all points $\alpha \in S^{d-1}$ with fixed height t.

So, the RHS essentially gives the contribution given by the S^{d-2} "circle" at a given height t to the integral $\int_{\xi \in S^{d-1}} L_{n,d}(\xi)\, dS^{d-1}(\xi)$.

As for the LHS, I'm less able to visualize what it represents. However, I notice that it appears to be a "discretized" version of the "inner product", divided by the dimension. This gives an "average" of these values.

Proof of Thm 2.9:

First, note that since $\{Y_{n,j} : 1 \le j \le N_{n,d}\}$ is an orthonormal basis of $\mathbb{Y}_n^d$, then for any $A \in \mathbb{O}^d$, indexing element $1 \le l \le N_{n,d}$, and $Y_{n,l}(A\xi) \in \mathbb{Y}_n^d$, we have

$$Y_{n,l}(A\xi) = \sum_{j=1}^{N_{n,d}} c_{lj} Y_{n,j}(\xi) \quad \text{where} \quad c_{lj} \in \mathbb{C}. \qquad (*)$$

Now, since we have an orthonormal basis,

$$\int_{S^{d-1}} Y_{n,j}(A\xi)\, \overline{Y_{n,k}(A\xi)}\, dS^{d-1}(\xi) = \int_{S^{d-1}} Y_{n,j}(\eta)\, \overline{Y_{n,k}(\eta)}\, dS^{d-1}(\eta) = \delta_{jk}.$$

So,

$$\begin{aligned}
\delta_{jk} &= \int_{S^{d-1}} Y_{n,j}(A\xi)\, \overline{Y_{n,k}(A\xi)}\, dS^{d-1}(\xi) \\
&= \int_{S^{d-1}} \left(\sum_{l=1}^{N_{n,d}} c_{jl} Y_{n,l}(\xi) \right) \overline{\left(\sum_{m=1}^{N_{n,d}} c_{km} Y_{n,m}(\xi) \right)} dS^{d-1}(\xi) \\
&= \sum_{l,m=1}^{N_{n,d}} c_{jl} \overline{c_{km}} \left[\int_{S^{d-1}} Y_{n,l}(\xi) \cdot \overline{Y_{n,m}(\xi)}\, dS^{d-1}(\xi) \right] \qquad [\dots] = \delta_{lm} \\
&= \sum_{l=1}^{N_{n,d}} c_{jl} \overline{c_{kl}}.
\end{aligned}$$

Hence $c_{jl}\overline{c_{kl}} = 1$ iff $j = k \Rightarrow CC^H = C\overline{C}^T = I$ where $C := [c_{jl}]$.

And so, $\displaystyle\sum_{j=1}^{N_{n,d}} \overline{c_{jl}}\, c_{jk} = \delta_{lk}$ for $1 \le k, l \le N_{n,d}$. $\quad (**)$

Now consider the sum

$$Y(\xi, \eta) := \sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\, \overline{Y_{n,j}(\eta)} \quad \text{for } \xi, \eta \in S^{d-1}.$$

For any $A \in \mathbb{O}^d$, we use $(*)$ to see that

$$Y(A\xi, A\eta) = \sum_{j=1}^{N_{n,d}} Y_{n,j}(A\xi)\, \overline{Y_{n,j}(A\eta)} = \sum_{j,k,l=1}^{N_{n,d}} c_{jk} \overline{c_{jl}}\, Y_{n,k}(\xi)\, \overline{Y_{n,l}(\eta)}.$$

Furthermore, from $(**)$, we have

$$= \sum_{k=1}^{N_{n,d}} Y_{n,k}(\xi)\, \overline{Y_{n,k}(\eta)} = Y(\xi, \eta).$$

Hence, $Y(A\xi, A\eta) = Y(\xi, \eta)$.