# Notes: Spherical Harmonics and Approximations on the Unit Sphere: An Introduction

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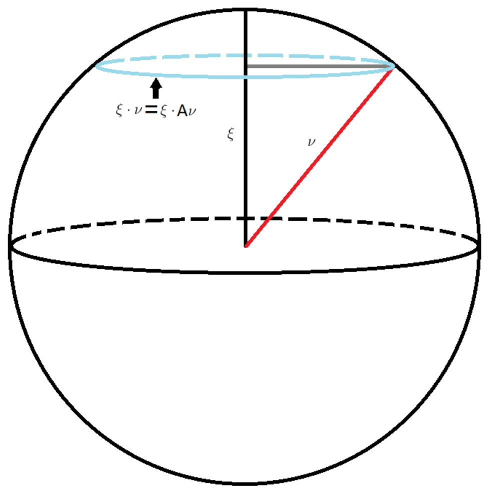
15/08/2018

# 1 Notation

**Definition 1.** $\mathbb{O}^d$-The set of all real orthogonal matrices of order $d$. A matrix $A \in \mathbb{R}^{d\times d}$ is orthogonal if $A^T A = I$ an extension of this is that $\mathbb{O}^d(\nu) := \{A \in \mathbb{O} : A\nu = \nu\}$.

**Definition 2.** $f_A$-If $f : \mathbb{R}^d \to \mathbb{C}$ and $A \in \mathbb{R}^{d\times d}$ then $f_A(x) := f(Ax)$. Note the primary use of this notation is for the study of symmetries of the given function. An example of this is a function $f$ satisfying $f = f_A$ for all $A \in \mathbb{O}^d(\xi)$ would then be constant on the blue circle on the picture below.




**Definition 3.** $\mathbb{H}_n^d$-the space of all homogeneous complex polynomials of degree $n$ in $d$ dimensions. Specifically, $\mathbb{H}_n^d := \{\sum_{\|\alpha\|=n} a_\alpha \mathbf{x}^\alpha\}$, noting that the dimension of this space is $\binom{n+d-1}{n}$ by a simple counting argument.

# 2 Spherical Harmonics

**Lemma 1. Harmonic Fact** If $\Delta f = 0$ then necessarily $\Delta f_A = 0$

**Definition 4. Invariant Subspace**-A general subspace $\mathbb{V}$ of functions defined in $\mathbb{R}^d$ or over a subset of $\mathbb{R}^d$ is *invariant* if for all $f \in \mathbb{V}$ and $A \in \mathbb{O}^d$ we have $f_A \in \mathbb{V}$.

**Definition 5. Reducible Subspace**-A general subspace $\mathbb{V}$ of functions defined in $\mathbb{R}^d$ or over a subset of $\mathbb{R}^d$ is *reducible* if $\exists \mathbb{V}_1, \mathbb{V}_2 \subseteq \mathbb{V}$ that satisfy $\mathbb{V} = \mathbb{V}_1 + \mathbb{V}_2$, $\mathbb{V}_1 \neq \emptyset \neq \mathbb{V}_2$, $\mathbb{V}_1$ and $\mathbb{V}_2$ are invariant, and $\mathbb{V}_1 \perp \mathbb{V}_2$. If no such $\mathbb{V}_1$ and $\mathbb{V}_2$ exist then $\mathbb{V}$ is called *irreducible.*

**Definition 6. Primitive Subspace**-A general subspace $\mathbb{V}$ of functions defined in $\mathbb{R}^d$ or over a subset of $\mathbb{R}^d$ is *primitive* if it is both invariant and irreducible.

## 2.1 Spaces of Homogeneous Polynomials

**Definition 7. Inner Product (on $\mathbb{H}_n^d$)**-An appropriate inner product on $\mathbb{H}_n^d$ is given by $(H_1, H_2) := H_1(\nabla)\overline{H_2(x)}$. Note that the parenthesis are used in the text to refer to inner products on a dual space.

**Definition 8.** $\mathbb{Y}_n(\mathbb{R}^d)$-These are the homogeneous harmonic polynomials of degree $n$ on $d$ variables. By definition $\mathbb{Y}_n(\mathbb{R}^d) \subseteq \mathbb{H}_n^d$. The number $N_{n,d}$ which is used frequently is defined to be the dimension of the space of this space. A straightforward arguement gives the relation

$$N_{n,d} = \text{dim}\mathbb{H}_{n-1}^d + \text{dim}\mathbb{H}_{n-1}^{d-1} \implies N_{n,d} = \frac{(2n+d-2)(n+d-3)}{n!(d-2)!}$$

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**Example 1** (2.6)**.** For $n = 0, 1$ we have $\mathbb{Y}_n(\mathbb{R}^d) = \mathbb{H}_n^d$

**Example 2** (2.6)**.** For $d = 3$ and $\forall \theta \in \mathbb{R}$ we have $(x_3 + ix_1 \sin\theta + ix_2 \cos\theta)^n \in \mathbb{Y}_n(\mathbb{R}^3)$. This is pretty simple as

$$\Delta\big([i\sin\theta, icos\theta, 1]x\big)^n = (n-2)\big([i\sin\theta, icos\theta, 1]x\big)^{n-2}\big|[i\sin\theta, icos\theta, 1]\big|^2 = 0$$

Alternatively, it can be seen using the multinomial theorem as follows here.

**Definition 9. Legendre Harmonic/Polynomial**-Mathworld-The Legendre harmonic of degree $n$ in $d$ dimensions, $L_{n,d} : \mathbb{R}^d \to \mathbb{R}$ is defined by the following conditions

- $L_{n,d} \in \mathbb{Y}_n(\mathbb{R}^d)$
- $L_{n,d}(Ax) = L_{n,d}(x) \quad \forall A \in \mathbb{O}^d(e_d), \forall x \in \mathbb{R}^d$
- $L_{n,d}(e_d) = 1$

These give the formula for the Legendre Harmonic ($L_{n,d}$) and Legendre Polynomial ($P_{n,d}$) as

$$\begin{aligned} L_{n,d}(x) &= n!\Gamma\left(\frac{d-1}{2}\right)\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{|x_{(d-1)}|^{2k}(x_d)^{n-2k}}{4^k k!(n-2k)!\Gamma(k+\frac{d-1}{2})} \\ P_{n,d}(t) &{:=} L_{n,d}(\xi_{(d)}) \qquad \text{Which is to say} \\ P_{n,d}(t) &= n!\Gamma\left(\frac{d-1}{2}\right)\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{(1-t^2)^{2k}t^{n-2k}}{4^k k!(n-2k)!\Gamma(k+\frac{d-1}{2})} \end{aligned}$$

**Definition 10. Chebyshev Polynomial of the First Kind**-Mathworld-These are given by the recurrence relation:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

Can be defined as the polynomials satisfying

$$T_n(x) = \left\{ \begin{array}{lr} \cos(n \arccos x), & \text{if } |x| \leq 1 \\ \cosh(n \arccos x), & \text{if } x \geq 1 \\ (-1)^n \cosh(n \text{ arccosh(-x)}), & \text{if } x \leq -1 \end{array} \right\}$$

**Definition 11. Dirichlet Kernel**-The collection of functions

$$D_n(x) := \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n}\cos(kx) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$$

**Definition 12. Reproducing Kernel Hilbert Space**-Let $H$ be a Hilbert space over $X$, $L_x \in H$ be defined by $L_x : f \to f(x)$ for all $f \in H$. Then $H$ is a Reproducing Kernel Hilbert space if, for all $x \in X$, $L_x$ is continuous ($\|L_x\|_H < M$).

The Reproducing Kernel is given by observing that $\exists K_x \in H$ such that for all $f \in H$, $f(x) = \langle f, K_x \rangle$. Then the Reproducing kernel is $K : X \times X \to \mathbb{R}$ given by

$$K(x, y) := \langle K_x, K_y \rangle$$

**Theorem 2.1.** Let $Y_n \in \mathbb{Y}_n^d$ and $\xi, \in \mathbb{S}^{d-1}$. Then $Y_n$ is invariant with respect to $\mathbb{O}^d(\xi)$ if and only if $Y_n(\nu) = Y_n(\xi)P_{n,d}(\xi \cdot \nu)$, $\forall \nu \in \mathbb{S}^{d-1}$

**Theorem 2.2. Addition Theorem** Let $\{Y_n, j :\leq j \leq N_{n,d}\}$ be an orthonormal basis of $\mathbb{Y}_n^d$ (WRT the $L^2$-norm). Then

$$\sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\overline{Y_{n,j}(\eta)} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}P_{n,d}(\xi \cdot \eta) \quad \forall \xi, \eta \in \mathbb{S}^{d-1}$$

As a consequence of the Addition Theorem we see that $K_{n,d}(\xi, \eta) := \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}P_{n,d}(\xi \cdot \eta)$ is the reproducing kernel of $\mathbb{Y}_n^d$

$\Delta f = 0 \implies \Delta f_A = 0$

*Proof.* First let $M \in \mathbb{O}^n$, $\Delta u = 0$, and $\{e_i\}$ be the standard basis on $\mathbb{R}^n$ (if $\delta_{i,j}$ is the Kronocker delta then $e_i = (\delta_{1,i}, \delta_{2,i}, ..., \delta_{n,i})$. Then

$$\begin{aligned}
\Delta u(Mx+c) &= \sum_i \frac{\partial^2}{\partial x_i^2} u(Mx+c) \\
&= \sum_i \frac{\partial}{\partial x_i} \sum_j u_j(Mx+c) \frac{\partial e_j(Mx+c)}{\partial x_i} \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} u_j(Mx+c) e_j M e_i^T \\
&= \sum_{i,j,k} u_{jk}(Mx+c) e_j M e_i^T e_k M e_i^T \\
&= \sum_{i,j,k} u_{jk}(Mx+c) e_j M e_i^T e_i M^T e_k^T && \text{Since } e_k M e_i^T \in \mathbb{R} \\
&= \sum_{j,k} u_{jk}(Mx+c) e_j M \left( \sum_i e_i^T e_i \right) M^T e_k^T \\
&= \sum_{j,k} u_{jk}(Mx+c) e_j M I M^T e_k^T \\
&= \sum_{j,k} u_{jk}(Mx+c) e_j e_k^T && \text{Since } M \in \mathbb{O}^n \text{ we have } MM^T = I \\
&= \sum_{j,k} u_{jk}(Mx+c) \delta_{i,k} \\
&= \sum_j u_{jj}(Mx+c) \\
&= \Delta u(Mx+c) = 0
\end{aligned}$$

□

**Statement**

*Proof.* First note that the original equation can be expressed in multi-index notation using $v = [i\sin\theta, i\cos\theta, 1]^T$ and $\{e_j\}$ to be the standard basis of $\mathbb{R}^3$ (Column Vectors),

$$\sum_{|\alpha|=n} \binom{n}{\alpha} v^\alpha x^\alpha$$

This gives the following

$$\begin{aligned}
\Delta \sum_{|\alpha|=n} \binom{n}{\alpha} v^\alpha x^\alpha &= \sum_{j=1}^{3} \sum_{|\alpha|=n} \binom{n}{\alpha} v^\alpha x^{\alpha-2e_j} (\alpha \cdot e_j)(\alpha \cdot e_j - 1) \\
&= \sum_{j=1}^{3} \sum_{|\alpha|=n} \binom{n}{\alpha - 2e_j} v^\alpha x^{\alpha-2e_j} \\
&= \sum_{j=1}^{3} \sum_{|\alpha|=n-2} \binom{n}{\alpha} v^{\alpha+2e_j} x^\alpha \\
&= \sum_{|\alpha|=n-2} \binom{n}{\alpha} v^\alpha x^\alpha \sum_{j=1}^{3} v^{2e_j} \\
&= \sum_{|\alpha|=n-2} \binom{n}{\alpha} v^\alpha x^\alpha (v \cdot v) \\
&= \sum_{|\alpha|=n-2} \binom{n}{\alpha} v^\alpha x^\alpha (-\sin^2\theta - \cos^2\theta + 1) = 0
\end{aligned}$$

As you can see from this as long $v \cdot v = 0$ this generalizes to examples in $\mathbb{Y}_d(\mathbb{R}^n)$. A nice fun example is $v = \left[\omega_d^0, \omega_d^{1/2}, ..., \omega_d^{(d-1)/2}\right]^T$ where $\omega_d$ is a primitive $d^{\text{th}}$ root of unity. $\square$

**Addition Theorem**

*Proof.* Let $A \in \mathbb{O}^d$ and $1 \leq k \leq N_{n,d}$, $Y_{n,k}(A\eta) \in \mathbb{Y}_n^d$ and then write in terms of the basis elements

$$Y_{n,k}(A\eta) = \sum_{j=1}^{N_{n,d}} c_{kj} Y_{n,j}(\eta), \quad c_{k,n} \in \mathbb{C}$$

This gives a nice way of seeing that

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} Y_{n,k}(Ax)\overline{Y_{n,j}(Ax)}\, dS^{d-1}(x) &= \int_{\mathbb{S}^{d-1}} \sum_{a,b=1}^{N_{n,d}} c_{ka}\overline{c_{jb}} Y_{n,a}(x)\overline{Y_{n,b}(x)}\, dS^{d-1}(x) \\
&= \sum_{a,b=1}^{N_{n,d}} c_{ka}\overline{c_{jb}} \int_{\mathbb{S}^{d-1}} Y_{n,a}(x)\overline{Y_{n,b}(x)}\, dS^{d-1}(x) \\
&= \sum_{a=1}^{N_{n,d}} c_{ka}\overline{c_{ja}} \\
\int_{\mathbb{S}^{d-1}} Y_{n,k}(Ax)\overline{Y_{n,j}(Ax)}\, dS^{d-1}(x) &= \int_{\mathbb{S}^{d-1}} Y_{n,k}(x)\overline{Y_{n,j}(x)} |\det(A)|\, dS^{d-1}(x) \\
&= \delta_{j,k} \\
\delta_{j,k} &= \sum_{a=1}^{N_{n,d}} c_{ka}\overline{c_{ja}}
\end{aligned}$$

Note that this last equality means $CC^H = C^H C = I$. Defining $Y(\xi, \eta) := \sum_{j=1}^{N_{n,d}} Y_{n,j}(\xi)\overline{Y_{n,j}(\eta)}$ we get

$$\begin{aligned}
Y(A\xi, A\eta) &= \sum_{j=1}^{N_{n,d}} \sum_{a,b=1}^{N_{n,d}} c_{ja}\overline{c_{jb}} Y_{n,a}(\xi)\overline{Y_{n,b}(\eta)} \\
&= \sum_{a,b=1}^{N_{n,d}} Y_{n,a}(\xi)\overline{Y_{n,b}(\eta)} \sum_{j=1}^{N_{n,d}} c_{ja}\overline{c_{jb}} \\
&= \sum_{a,b=1}^{N_{n,d}} Y_{n,a}(\xi)\overline{Y_{n,b}(\eta)}\delta_{a,b} \\
&= \sum_{a=1}^{N_{n,d}} Y_{n,a}(\xi)\overline{Y_{n,a}(\eta)} = Y(\xi, \eta)
\end{aligned}$$

One of the middle steps can be seen by noting $CC^H = I$. The result is that $Y(\xi, \eta)$ is rotation invariant. Fixing $\xi$ gives $Y(\xi, \cdot) \in \mathbb{Y}_n^d$ and is invariant under $A \in \mathbb{O}^d(\xi)$ (remembering $A\xi = \xi$ makes this clear). By Theorem 2.1 $Y(\xi, \nu) = Y(\xi, \xi)P_{n,d}(\xi \cdot \nu)$. Now if $P_{n,d}(\xi \cdot \nu) \neq 0$ (then using continuity we see $\forall \xi, \nu \in \mathbb{S}^{d-1}$)

$$Y(\xi, \xi) = \frac{Y(\xi, \nu)}{P_{n,d}(\xi \cdot \nu)} = Y(\nu, \nu)$$

To find out what this constant is

$$Y(\xi,\xi) = \sum_{j=1}^{N_{n,d}} \left|Y_{n,j}(\xi)\right|^2$$

$$Y(\xi,\xi) = \fint_{\mathbb{S}^{d-1}} Y(\xi,\xi)\, dS^{d-1}(\xi) = \fint_{\mathbb{S}^{d-1}} \sum_{j=1}^{N_{n,d}} \left|Y_{n,j}(\xi)\right|^2 dS^{d-1}(\xi) = \sum_{j=1}^{N_{n,d}} \frac{1}{|\mathbb{S}^{d-1}|} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}$$

Finally, Substituting it back in we get

$$Y(\xi,\eta) = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} P_{n,d}(\xi \cdot \eta)$$

□